Def: **Set** is the collection of well defined elts. or objects.

* Generally the sets are denoted by A, B, C, ... described in two ways ;

i) Tabular (ii) set builder.

Ex $A = \{1,2,3,4,5\} = \{x \mid x \in N, 1 \le x \le 5\}$

$B = \{a,e,i,o,u\} = \{x \mid x$ is the vowel of english alphabet$\}$

$N^+, Z^+, I^+ = \{1,2,3,4,5,\dots\} \Rightarrow$ the set of natural nos. or +ve integers.

$N^+, W = \{0,1,2,3,\dots\} \Rightarrow$ The set of whole nos.

$Z, I = \{0, \pm 1, \pm 2, \pm 3 \dots\} \Rightarrow$ The set of Integers.

$Q = \{x \mid x = \frac{p}{q}, q \neq 0\} \Rightarrow$ The set of rational nos.

$Q', R-Q = \{x \mid x \in R, x \notin Q\} \Rightarrow$ The set of irrational nos.

$R = Q \cup Q' \Rightarrow$ The set of real nos.

$C = \{z \mid z = x+iy, \forall x,y \in R\} \Rightarrow$ The set of complex no.

Prime :→

The no. $P > 1$ is said to be a prime no. if it has no divisors. The nos. are 2, 3, 5, 7, 11, 13, 17, 19, ...

Composite no. :→

The no. $P > 1$ which is not a prime no. is called as a composite no.

Divisibility :→

Let $a, b \in N$

$a \mid b$ reads as a divides b

or a is the divisible of b.

or b is the " by a

iff there exist $k \in Z$ for which $\boxed{b = ka}$

Ex * $2|6$ as $6 = 3\times 2$; $3 \in \mathbb{Z}$

* $5|7$ is wrong as $7 = (1.4)\times 5$ but $1.4 \notin \mathbb{Z}$
i.e $5 \nmid 7$

* $a|a$, $a \neq 0$

* If $a|b$ and $b|a$ then $b \neq a$

Pf. $a|b \Leftrightarrow b = k_1 a$, $k_1 \in \mathbb{Z}$

Now $b|a \Leftrightarrow a = k_2 b$, $k_2 \in \mathbb{Z}$

Consider $b = k_1 a = k_1(k_2 b) \Rightarrow b = k_1 k_2 b$
$\Rightarrow k_1 k_2 = 1$
$\Rightarrow k_1 = k_2 = \pm 1$
$\Rightarrow \boxed{b = \pm a}$

* If $a|b$ & $b|c$, then $a|c$

Pf. $a|b \Leftrightarrow b = k_1 a$; $k_1 \in \mathbb{Z}$
& $b|c \Leftrightarrow c = k_2 b$; $k_2 \in \mathbb{Z}$
Consider $c = k_2 b = (k_2 k_1) a = k_3 a$
$\Rightarrow c = k_3 a$
$\Rightarrow a|c$ ☐

Relatively prime :→

Let $a, b \in \mathbb{N}$; a and b are said to be relatively prime if their g.c.d is 1 i.e $g.c.d(a,b) = 1$.
$\Leftrightarrow$ a and b are relatively prime to each other.

Congruence Modulo :→

Let $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$; $a \equiv b \pmod n$
reads as "a congruent to b modulo n"
if $b-a$ is divisible by n or $n | b-a$.

Ex
$2 \not\equiv 3 \pmod 5$
$5 \equiv 3 \pmod 2$
$3 \equiv 8 \pmod 5$
$8 \equiv 5 \pmod 3$

* $a \equiv a \pmod n$

Pf. Since $n \in \mathbb{N}$ Always $n|0$ divisible by any natural nos.

$\Rightarrow n \mid a-a$

$\Rightarrow a \equiv a \pmod n$

* If $a \equiv b \pmod n$ then $b \equiv a \pmod n$

Pf. $a \equiv b \pmod n$

i.e $n \mid b-a$

$\Rightarrow n \mid -(b-a)$

$\Rightarrow n \mid a-b$

$\Rightarrow b \equiv a \pmod n$

* If $a \equiv b \pmod n$ & $b \equiv c \pmod n$ then $a \equiv c \pmod n$

Pf. $a \equiv b \pmod n$

$\Rightarrow n \mid b-a \Rightarrow b-a = k_1 n \;;\; k_1 \in \mathbb{Z}$

& $b \equiv c \pmod n$

$\Rightarrow n \mid c-b \Rightarrow c-b = k_2 n \;;\; k_2 \in \mathbb{Z}$

Now $(b-a)+(c-b) = k_1 n + k_2 n$

$\Rightarrow c-a = (k_1+k_2) n$

$\Rightarrow c-a = k_3 n \;;\; k_3 \in \mathbb{Z}$

$\Rightarrow n \mid c-a$

$\Rightarrow \boxed{a \equiv c \pmod n}$

Set operation :→

$A \cup B = \{x / x \in A \vee x \in B\}$

$A \cap B = \{x / x \in A \wedge x \in B\}$

$A - B = \{x / x \in A \wedge x \notin B\}$

$A' = X - A$

* $A \times B = \{(x,y) \mid x \in A, y \in B\}$

Ex $A = \{1,2,3\}$, $B = \{3,4,5\}$

$A \cup B = \{1,2,3,4,5\}$

$A \cap B = \{3\}$

$A - B = \{1,2\}$

$A \times B = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5)\}$

* $A \subset B$ i.e $x \in A \Rightarrow x \in B$ A B

* The sets of all subsets of A is P(A).
* If $|A| = n$, then $|P(A)| = 2^n$.

Pf: The no. of subsets contains '0' elt. of set A = $C(n,0)$

" " " " '1' " " $= C(n,1)$

" " " " '2' " " $= C(n,2)$

⋮

" " " " 'n' " " $= C(n,n)$

Total subset $= C(n,0) + C(n,1) + \cdots + C(n,n)$

$= 2^n$

Ex
* $A = \{\}$ $\rightarrow |A| = 0$

  $P(A) = \{\emptyset\} \Rightarrow |P(A)| = 2^0 = 1$
* $A = \{a\}$ $\rightarrow |A| = 1$

  $P(A) = \{\emptyset, \{a\}\} \Rightarrow |P(A)| = 2^1 = 2$
* $A = \{a, b\}$ $\rightarrow |A| = 2$

  $P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \Rightarrow |P(A)| = 2^2 = 4$

Q* What is the diff. betn similar sets & equal sets.

Equal sets	similar sets
$A \subset B + B \subset A$	$A = B$
$\Leftrightarrow A = B$	iff $\lvert A \rvert = \lvert B \rvert$

Thm :→
* $|A \cup B| = |A| + |B| - |A \cap B|$
* Two sets A and B are disjoint iff $A \cap B = \emptyset$

  So $|A \cup B| = |A| + |B|$
* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Relation :→

* (a, b) denotes an ordered pair where $(a,b) \neq (b,a)$

  $(a,b) = (x,y) \Leftrightarrow a = x$ & $b = y$
* Let A and B are two non-empty sets.

  $|A| = m$, $|B| = n$

  So $|A \times B| = mn$
* Any subset of $A \times B$ is said to be a binary relation for set A to B.

* R be a binary rel^n from set A to B i.e $R \subseteq A \times B$
* If $|A| = m$ and $|B| = n$ then there will be 2^{mn} no. of binary relation.

Ex $A = \{1, 2\}$
$B = \{a, b\}$

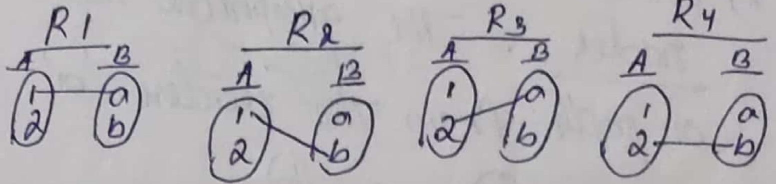
The total no. of binary relation exists from set A to B.

$|A| = 2$ & $|B| = 2$, $|A \times B| = 2 \times 2 = 4$

$|P(A \times B)| = 2^4 = 16$

Here $A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$

$R_1 = \{(1,a)\}$
$R_2 = \{(1,b)\}$
$R_3 = \{(2,a)\}$
$R_4 = \{(2,b)\}$



Notes :→

* R be a rel^n on set A
  so $R \subseteq A \times A$
* If $|A| = n$ then there will be 2^{n^2} no. of binary rel^n exist on set A. Ex Let $A = \{1, 2, 3, 4, 5, 6\}$

(1) $R = \{(j, k) \mid j \text{ divides } k, \forall j, k \in A\}$

$R \subseteq A \times A$

$(j, k) \in R \Leftrightarrow j \mid k$

Since $A \times A = \{(1,1), (1,2), (1,3), \ldots, (1,6)$
$(2,1), \ldots, (2,6)$
$(3,1), \ldots, (3,6)$
$(4,1), \ldots, (4,6)$
$(5,1), \ldots, (5,6)$
$(6,1), \ldots, (6,6)\}$

So $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$

(2) $R = \{(x, y) \mid x \,\&\, y$ are relatively prime to eachother $\forall x, y \in A\}$

Here $R \subseteq A \times A$

$(x, y) \in R$ iff $g.c.d(x, y) = 1$

So $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1), (4,3), (4,5), (5,1), (5,2), (5,3), (5,4), (5,6), (6,1), (6,5)\}$.

(4) $R = \{(x,y) \mid (x-y)^2 \in A, \forall x,y \in A\}$

$= \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (3,5), (4,2), (4,3), (4,5), (4,6), (5,3), (5,4), (5,6), (6,4), (6,5)$

(5) $R = \{(x,y) \mid \frac{x}{y} \in A; \forall x, \forall y \in A\}$

$= \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4), (5,1), (5,5), (6,1), (6,2), (6,3), (6,6)\}$

<u>Graphical Representation</u> :→

*) (a,b) be the ordered pair where 'a' and 'b' are said to be <u>nodes</u>. The graphical meaning of (a,b) is there exists a path from the vertex 'a' to the vertex 'b'.

$$a \longrightarrow b$$

* (a,a) represents a <u>self loop</u>.

* $\{a,b\}$ be the path betn 'a' & 'b' $\quad a \text{ --- } b$

$\{a,b\} = \{b,a\}$

* (a,b) be the directed path / edge.

$\{a,a\}$ be the non-directed path / edge.

So $G = (V, E)$ be a graph where 'V' be the <u>vertex</u> set and E is the <u>edge</u>.

⇒ If the edges contained in the graph are <u>directed</u> then the graph is said to be <u>directed graph (or) Digraph</u>.

<u>Note</u> The ~~binary relat~~ graphical representation of a binary relation defined on single set A be <u>digraph</u>.

⇒ If the edges contained in the graph are undirected then the graph is a <u>non-directed graph</u>.

<u>Q</u> Construct the digraphs of the following sets;

Let $A = \{1,2,3,4\}$

(1) $R = \{(x,y) \mid x+y = \text{even}; \forall x, \forall y \in A\}$

$R \subseteq A \times A$; $A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

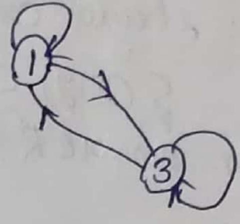
(3,15)
(6,5)

where $R = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$

(2) $R = \{(x,y) \mid x \equiv y \pmod 2, \forall x, y \in A\}$

Here $R \subseteq A \times A$

$R = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$



* Let R be a relation defined on set A

so $R \subseteq A \times A$.

$\Rightarrow \forall a \in A$, if $(a,a) \in R$ then R is reflexive.

$\Rightarrow \forall a, b \in A$, for $(a,b) \in R$ then $(b,a) \in R$ then R is symmetric.

Reflexive :→

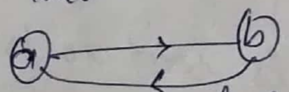
If every vertex contains self loop then R is said to be reflexive.

(OR) A relⁿ R on a set A is reflexive if aRa for every $a \in A$ that is; $\forall a \in A$, if $(a,a) \in R$ then R is reflexive.

Ex If $R_1 = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ be a relⁿ on $A = \{1,2,3\}$ then R_1 is reflexive relⁿ since $a \in A$, $(a,a) \in R$.

Symmetric :→

If there exists any edge or path from vertex A to vertex B, then their should be the reversed path.

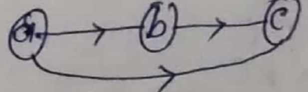


(OR) A relⁿ R on a set A is symmetric if whenever $aRb \Rightarrow bRa$, i.e if $(a,b) \in R$ then $(b,a) \in R$

Ex $R = \{(x,y) \in R^2 : x^2+y^2 = 1\}$ is a symmetric relⁿ on R since if $(x,y) \in R$ then $(y,x) \in R$ too i.e if $x^2+y^2 = 1$ then $y^2+x^2 = 1$

Transitive :→

$\forall a, b, c \in A$ if $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$ then R is said to be transitive.



(OR) A rel^n R on a set A is transitive if $(a,b) \in R$ & (b,c) then $(a,c) \in R$ i.e aRb & $bRC \Rightarrow aRC$.

Ex The rel^n "is less than" & "is greater than" are transitive rel^ns on the set of real nos. If $a<b$ & $b<c$ implies $a<c$ & if $a>b$ & $b>c$ implies $a>c$ for all real nos. a,b,c.

Irreflexive :→

R is said to be irreflexive, $\forall a \in A$ iff $(a,a) \notin R$.

(OR) The diagram should contain no self rule.

Ex The rel^n $R = \{(x,y) \in R : x<y\}$ is an irreflexive rel^n since $x<x$ for no $x \in R$.

Antisymmetric :→

R is said to be antisymmetric if for $(a,b) \in R$ & $(b,a) \in R$ then $a=b$. ex $\left.\begin{matrix} A \subset B \\ B \subset A \end{matrix}\right\} \to A=B$ $\left.\begin{matrix} a<b \\ b<a \end{matrix}\right\} \Rightarrow a=b$ $\left.\begin{matrix} a|b \\ \& \ b|a \end{matrix}\right\} \Rightarrow a=b$

OR There should not be any path in between two vertices in both direction.

OR self loop are permitted here.

Asymmetric :-

R is said to be asymetric if for $(a,b) \in R$ then $(b,a) \notin R$ If there exist a path from vertex A to vertex B then their should not be the reversed path and again self loops are not allowed.

POSET :- If the rel^n R satisfies reflexive, antisymmetric, and transitive properties then R is said to be a partial ordered rel^n on set A.

→ In such case the set A is said to be a partial ordered set (POS).

→ [A, R] is said to be a POSET.

Note: Find the transitive of the def^n of the transitive property.

Sol^n If aRb & bRC then aRC

P: aRb & bRC

$\overline{P}$: $a \not R b$ & $b \not R C$

Q: aRC

$\overline{Q}$: $a \not R C$

$P \to Q$

Converse : $Q \to P$

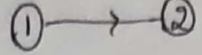
Inverse : $\overline{P} \to \overline{Q}$

Contrapositive: $\overline{Q} \to \overline{P}$

$\boxed{(P \to Q) \equiv (\overline{Q} \to \overline{P})}$

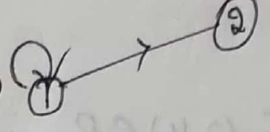
i.e contrapositive :→
If $(a,c) \notin R$ then neither $(a,b) \in R$ nor $(b,c) \in R$.

Ex ① $A = \{1,2\}$



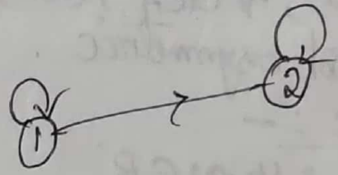
→ $R_1 = \{(1,2)\}$
→ R_1 is not reflexive.
As $(1,1) \notin R$ and $(2,2) \notin R$ as $1,2 \in A$
* R_1 is irreflexive.
* R_1 is not symmetric as $(1,2) \in R$ but $(2,1) \notin R$
* R_1 is transitive.
* R_1 is antisymmetric.
* R_1 is assymetric.

Ex (2) $A = \{1,2\}$
$R_2 = \{(1,1),(1,2)\}$



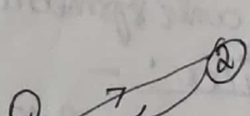
* R_2 is not reflexive as $2 \in A$ but $(2,2) \notin R$
* R_2 is not irreflexive as $1 \in A$ for which $(1,1) \in A$.
  R_2 is not symmetric.
  as $(1,2) \in R$ but $(2,1) \notin R$.
* R_2 is transitive.
* R_2 is antisymmetric.
* R_2 is not assymetric

Ex ③ $A = \{1,2\}$
$R_3 = \{(1,1),(1,2),(2,2)\}$



* R_3 is reflexive.
* R_3 is not irreflexive.
* R_3 is not symmetric.
* R_3 is transitive.
* R_3 is antisymmetric.
* R_3 is not assymetric.

Ex-④ $A = \{1,2\}$, $R_4 = \{(1,1),(1,2),(2,1)\}$



R_4 is not reflexive.
R_4 is not irreflexive one as there is one self loop.
R_4 is symmetric.
R_4 is not antisymmetric.
R_4 is not assymetric.
R_4 is not transitive.

Relations defined on Infinite set :→

Ex $R = \{(a,b) \mid a \text{ divides } b, \forall a,b \in Z^+\}$

Verification :→

$R \subseteq Z^+ \times Z^+$

$(a,b) \in R$ if $a|b$

* Reflexive :-
  - since every integer is divisible by itself.

    so $\forall a \in Z^+$, $a|a \Rightarrow (a,a) \in R$

    $\therefore$ R is reflexive.

* R is irreflexive.
* Symmetric :-

  Let $(a,b) \in R \Rightarrow a|b$ $\Rightarrow b \nmid a$ $\Rightarrow (b,a) \in R$ [$\because (2,4) \in R$ $\Rightarrow 2|4$ $\Rightarrow 4 = 2\times2$ but $(4,2) \in R$ bcz $4 \nmid 2$]

  $\therefore$ R is not symmetric.

* Asymmetric :-

  As $(a,b) \in R$ and $(b,a) \notin R$

  Also self loop, $\forall a \in A$, $(a,a) \in R$

  So R is not asymmetric.

* Antisymmetric :-

  Let $(a,b) \in R$ & $(b,a) \in R$

  $\Rightarrow a|b$ & $b|a$

  $\Rightarrow a = b$

  $\therefore$ R is antisymmetric.

* Transitive :-

  Let $(a,b) \in R$ & $(b,c) \in R$

  $\Rightarrow a|b$ & $b|c$

  $\Rightarrow a|c$ [$a|b \Rightarrow b = ak_1$, & $b|c \Rightarrow c = bk_2 = ak_1k_2 = ak_3 \Rightarrow c = ak_3 \Rightarrow a|c$]

  $\Rightarrow (a,c) \in R$

  $\Rightarrow$ R is transitive.

Thus, R satisfy reflexive, antisymmetric and transitive.

Hence R be a part of P.O.R and Z^+ be a P.O.S.

$[Z^+, R]$ be a POSET.

Ex $R = \{(a,b) \mid a-b = \text{even} ; \forall a,b \in \mathbb{Z}\}$

$\therefore R \subseteq \mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

* Reflexive :→
$\forall a \in \mathbb{Z}$; $a-a = 0 = \text{Even}$
$\Rightarrow (a,a) \in R \Rightarrow R$ is reflexive.

* Irreflexive : →
Since $(a,a) \in R \Rightarrow R$ is not irreflexive.

* Symmetric :→
Let $(a,b) \in R \Rightarrow a-b = \text{even} \Rightarrow b-a = \text{Even} \Rightarrow (b,a) \in R$
$\Rightarrow R$ is symmetric.

* Assymmetric :→
Since $(a,b) \in R$ and $(b,a) \in R \Rightarrow R$ is not assymmetric

* Transitive :→
Let $(a,b) \in R$ & $(b,c) \in R$
Let $(a,b) \in R$ & $(b,c) \in R$
$\Rightarrow a-b = \text{even}$ & $b-c = \text{even}$
$\therefore a-b+b-c = \text{even} + \text{even}$
$\Rightarrow a-c = \text{even}$
$\Rightarrow (a,c) \in R$
$\Rightarrow R$ is transitive.

* Antisymmetric :→
Since $(a,b) \in R$ & $(b,a) \in R$
$\Rightarrow a \neq b$
$\therefore R$ is not antisymmetric.

Thus R is equivalence rel^n on $\mathbb{Z}$.

Ex $R = \{(a,b) \mid a \equiv b \pmod 5) ; \forall a, b \in \mathbb{Z}\}$

Here $R \subseteq \mathbb{Z} \times \mathbb{Z}$
$(a,b) \in R \Leftrightarrow a \equiv b \pmod 5)$
i.e $5 \mid b-a$

Reflexive :-
Let $\forall a \in \mathbb{Z}$, since $5 \mid 0$
$\Leftrightarrow 5 \mid a-a$
$\Leftrightarrow a \equiv a \pmod 5)$
$\Leftrightarrow (a,a) \in R$
$\Leftrightarrow R$ is reflexive.

Irreflexive :-
Since $(a,a) \in R$, so R is not irreflexive.
symmetric :-
Let $(a,b) \in R \Rightarrow a \equiv b \pmod 5 \Rightarrow 5 \mid b-a \Rightarrow 5 \mid -(b-a)$
$\Rightarrow 5 \mid a-b$
$\Rightarrow b \equiv a \pmod 5$
$\Rightarrow (b,a) \in R$
$\Rightarrow$ R is symmetric.

* R is not assymmetric.
* R is not antisymmetric.
* Let $(a,b) \in R$ & $(b,c) \in R$
$\Rightarrow a \equiv b \pmod 5$ & $b \equiv c \pmod 5$
$\Rightarrow 5 \mid b-a$ — ① $\Rightarrow 5 \mid c-b$ — ②
From ① & ② $5 \mid (b-a)+(c-b)$
$\Rightarrow 5 \mid c-a$
$\Rightarrow a \equiv c \pmod 5$
$\Rightarrow (a,b) \in R$
$\Rightarrow$ R is transitive
Thus R be an equivalence reln over $\mathbb{Z}$.

Ex $R = \{(a,b) \mid \gcd(a,b) = 1, \forall a,b \in \mathbb{Z}^+\}$
$R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$
$(a,b) \in R$ iff $\gcd(a,b) = 1$
* since $\forall a \in \mathbb{Z}^+$ $g.c.d(a,a) = a \neq 1$
$\Rightarrow$ R is not reflexive.
* R is not irreflexive as $1 \in \mathbb{Z}^+$ for which $(1,1) \in R$.
* Let $(a,b) \in R \Rightarrow g.c.d(a,b) = 1 \Rightarrow \gcd(b,a) = 1 \Rightarrow (b,a) \in R$.
$\Rightarrow$ R is symmetric
* R is not assymetric.
* R is not antisymmetric.
* Let $(a,b) \in R$ + $(b,c) \in R \Rightarrow \left.\begin{array}{l}\gcd(a,b)=1 \\ \gcd(b,c)=1\end{array}\right] \Rightarrow \gcd(a,c) \neq 1$
$\Rightarrow (a,c) \notin R$
$\Rightarrow$ R is not transtv. $\left[\begin{array}{l}(2,5) \in R \text{ as } \gcd(2,5)=1 \\ (5,6) \in R \text{ as } \gcd(5,6)=1\end{array}\right.$
But $(2,6) \notin R$
$\gcd(2,6) = 2 \neq 1$

HW * (1) $R = \{(a,b) \mid a = b^2, \forall a,b \in \mathbb{Z}\}$
(2) $R = \{(a,b) \mid a < b; \forall a,b \in N\}$

Note (1) $A = \{a\}$ $\qquad R \subset P(A) \times P(A)$

$P(A) = \{\phi, \{a\}\}$

$P(A) \times P(A) = \{(\phi, \phi), (\phi, \{a\}), (\{a\}, \phi), (\{a\}, \{a\})\}$

so $R = \{(\phi, \phi), (\phi, \{a\}), (\{a\}, \{a\})\}$ 

(2) $A = \{a, b\}$; $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$P(A) \times P(A) = \{(\phi, \phi), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{a, b\}), (\{a\}, \phi), (\{a\}, \{a\}), (\{b\}, \phi), (\{b\}, \{a\}), (\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{a\}, \{b\}), (\{a\}, \{a, b\}), (\{a, b\}, \phi), (\{a, b\}, \{a\}), (\{a, b\}, \{b\}), (\{a, b\}, \{a, b\})\}$

$R = \{(\phi, \phi), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{a, b\}), (\{a\}, \{a\}), (\{a\}, \{b\}), \cancel{(\{a, b\}, \phi)} (\{a, b\}, \{a, b\}), (\{b\}, \{a, b\}), (\{a\}, \{a, b\}), (\{b\}, \{b\})\}$

(3) Q $A = \{a, b, c\}$,

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$R \subset P(A) \times P(A)$

$P(A) \times P(A) = ?$

Ex of $R = \{(x, y) \mid x \leq y, \forall x, y \in P(A)\}$ then find R in tabular form for $A = \{a\}$, $A = \{a, b\}$, $A = \{a, b, c\}$.

Sol Let A be any finite set.

$R = \{(x, y) \mid x \leq y ; \forall x, y \in P(A)\}$

Here $R \subseteq P(A) \times P(A)$

$(x, y) \in R \Leftrightarrow x \leq y$

Reflexive $\forall x \in P(A)$, every set is subset of itself

so $x \leq x \Rightarrow (x, x) \in R \Rightarrow R$ is reflexive.

Irreflexive

R is not irreflexive.

Symmetric Let $(x, y) \in R \Rightarrow x \leq y \Rightarrow y \not\leq x$

$\Rightarrow (y, x) \notin R$

$\Rightarrow R$ is not symmetric

Asymmetric

R is asymmetric as $(x, y) \in R$

Anti symmetric

Let $(x, y) \in R$ & $(y, x) \in R$

so $x \leq y$ & $y \leq x$

$\Rightarrow x = y \Rightarrow R$ is antisymmetric.

Transitive Let $(x, y) \in R$ & $(y, z) \in R$

$\Rightarrow x \leq y$ & $y \leq z$

$\Rightarrow x \leq z$

$\Rightarrow (x, z) \in R$ $\therefore R$ is trans

Then R be a P.O.R over $P(A)$.

$P(A)$ be a P.O.S

$[P(A), R]$ be a poset.

Scanned with CamScanner

Note The graphical representation of a POSET is said to be POSET diagram or Hasse diagram.

How to construct Hasse diagram :→

1) Construct the rel^n R in tabular form.
2) Delete all the identity pairs (x,x).
3) Delete the pair (x,z) if $(x,y) \in R$ and $(y,z) \in R$
4) Now construct the graph with the remaining element where edges are to be drawn from below to above without arrow mark.

Ex $A = \{a, b\}$

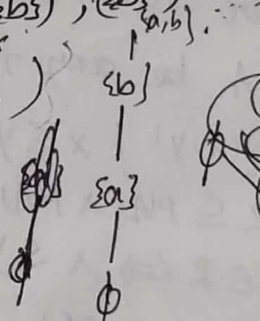
$P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$R \subseteq P(A) \times P(A)$

$P(A) \times P(A) = \{(\phi,\phi), (\phi,\{a\}), (\phi,\{b\}), (\phi,\{a,b\}), (\{a\},\phi), (\{a\},\{a\}), (\{b\},\{a\}), (\{b\},\{b\}), (\{b\},\{a,b\}), (\{a\},\{b\}), (\{a\},\{a,b\}), (\{b\},\phi), (\{b\},\{a\}), ~~(\{b\},\{a,b\})~~, (\{a,b\},\{a,b\})\}$, $(\{a,b\},\phi)$, $(\{a,b\},\{a\})$, $(\{a,b\},\{b\})$, ...

$R = \{(\phi,\phi), (\phi,\{a\}), (\phi,\{b\}), (\phi,\{a,b\}), (\{a\},\{a\}), (\{a\},\{b\}), (\{a\},\{a,b\}), (\{b\},\{b\}), (\{b\},\{a,b\}), (\{a,b\},\{a,b\})\}$

$R = \{(\phi,\{a\}), (\{a\},\{b\}), (\{b\},\{a,b\})\}$



Ex $A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$

$R \subseteq P(A) \times P(A) = \{(\phi,\phi), (\phi,\{a\}), (\phi,\{b\}), (\phi,\{c\}), (\phi,\{a,b\}), (\phi,\{b,c\}), (\phi,\{c,a\}), (\phi,\{a,b,c\}), (\{a\},\phi), (\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{a,b\}), (\{a\},\{b,c\}), (\{a\},\{c,a\}), (\{a\},\{a,b,c\}), (\{b\},\phi), (\{b\},\{a\}), (\{b\},\{b\}), (\{b\},\{c\}), (\{b\},\{a,b\}), (\{c\},\phi), (\{c\},\{a\}), (\{c\},\{b\}), (\{c\},\{c\}), (\{c\},\{a,b\}), (\{a,b\},\phi), \ldots\}$

$R = \{(\phi,\phi), (\phi,\{a\}), (\phi,\{b\}), (\phi,\{c\}), (\phi,\{a,b\}), (\phi,\{b,c\}), (\phi,\{c,a\}), (\phi,\{a,b,c\}), (\{b\},\{b\}), (\{a\},\{a\}), (\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{a,b\}), (\{a\},\{b,c\}), (\{a\},\{c,a\}), (\{a\},\{a,b,c\}), (\{c\},\{c\}), (\{c\},\{a,b\}), (\{b\},\{c\}), (\{b\},\{a,b\}), (\{b\},\{b,c\}), (\{b\},\{a,c\}), (\{b\},\{a,b,c\}), (\{c\},\{b,c\}), (\{c\},\{a,c\}), (\{c\},\{a,b,c\}), (\{a,b\},\{a,b\}), (\{a,b\},\{b,c\}), (\{a,b\},\{c,a\}), (\{a,b\},\{a,b,c\}), (\{b,c\},\{b,c\}), (\{b,c\},\{a,c\}), (\{b,c\},\{a,b,c\}), (\{c,a\},\{c,a\}), (\{c,a\},\{a,b,c\}), (\{a,b,c\},\{a,b,c\})\}$

$= \{(\phi,\{a\}), (\{a\},\{b\}), (\{b\},\{c\}), (\{c\},\{a,b\}), (\{a,b\},\{b,c\}), (\{b,c\},\{a,c\}), (\{a,c\},\{a,b,c\})\}$

$\phi - \{a\} - \{b\} - \{c\} - \{a,b\} - \{b,c\} - \{a,c\} - \{a,b,c\}$.

Ex Let $A = \{1,2,3,4,5\}$

$R = \{(x,y) \mid x \geq y, \forall x,y \in A\}$

Is R be a P.O.R. If yes draw the Hasse diagram of [A, R].

Sol^n $R \subseteq A \times A$ where $(x,y) \in R$ iff $x \geq y$

$A \times A = \{(1,1), \ldots (1,5), (2,1), \ldots (2,5) \ldots (5,5)\}$

$R = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), \ldots (5,5)\}$

R is reflex

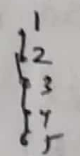
R is antisymmetric

R is transitive

R be a P.O.R

For Hasse diagram $R = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$

$R = \{(2,1), (3,2), (4,3), (5,4)\}$



Ex A be a set of divisors of 12

$R = \{(a,b) \mid a \text{ divides } b, \forall a,b \in A\}$

Here $A = \{1,2,3,4,6,12\}$

$(a,b) \in R \Leftrightarrow a \mid b$

$A \times A = \ldots$

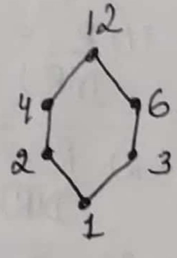
$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,12), (4,12), (6,12), (12,12)\}$

R is refl

R is antisym

R is tr

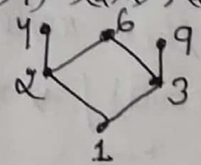
$R = \{(1,2), (1,3), (2,4), (2,6), (3,6), (4,12), (6,12)\}$



Ex Let $A = \{1,2,3,4,6,9\}$

$R = \{(a,b) \mid a \text{ divides } b; \forall a,b \in A\}$

$= \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (2,2), (2,4), (2,6), (3,3), (3,6), (3,9), (4,4), (6,6), (9,9)\}$



Ex 1 $A = \{1,2\}$

Smallest equi rel^n is $R = \{(1,1), (2,2)\}$

Largest equivalence " $R = A \times A$

2 $A = \{1,2,3\}$

Smallest equivalence rel^n is $R = \{(1,1), (2,2), (3,3)\}$

Largest equivalence rel^n is $R = A \times A$

__Inverse of a relation :-__

Let $R \subseteq A \times B$

It's inverse rel^n is R^{-1}

where $R^{-1} = \{(b,a) \mid \forall (a,b) \in R\}$

Ex-1 $A = \{1,2,3\}$, $B = \{a,b,c\}$

$R \subseteq A \times B$

$R = \{(1,a), (2,b), (2,c), (3,c)\}$

$R^{-1} = \{(a,1), (b,2), (c,2), (c,3)\}$

Ex-2 $A = \{1,2,3\}$

$R \subseteq A \times A$

$R = \{(1,1), (1,2), (2,3), (3,3)\}$

$R^{-1} = \{(1,1), (2,1), (3,2), (3,3)\}$

Note (1) If $R \subseteq A \times B$ then $R^{-1} \subseteq B \times A$

(2) $\bar{R}$ or R^c be the complement of the relⁿ R where

$\bar{R} = (A \times B) - R$.

Ex $A = \{1,2,3\}$, $B = \{a,b,c\}$

$A \times B = \{(1,a) (1,b), (1,c), (2,a) (2,b), (2,c), (3,a), (3,b), (3,c)\}$

Since $R = \{(1,a), (2,b), (2,c), (3,c)\}$

so $\bar{R} \in (A \times B) - R = \{(1,b), (1,c), (2,a), (3,a), (3,b)\}$

Note * Let $R \subseteq A \times B$

$D(R)$ denotes the domain of R.

$R(R)$ " " range of R.

where $D(R) = \{a \mid \forall (a,b) \in R\}$

$R(R) = \{b \mid \forall (a,b) \in R\}$

Ex $A = \{1,2,3\}$, $B = \{4,5,6\}$

(1) $R = \{(1,4), (1,6), (2,5), (3,6)\}$

$D(R) = \{1,2,3\}$; $R(R) = \{4,5,6\}$

(2) $R = \{(1,4), (1,5), (1,6)\}$

$D(R) = \{1\}$; $R(R) = \{4,5,6\}$

(3) $R = \{(1,4), (2,4), (3,4)\}$

$D(R) = \{1,2,3\}$, $R(R) = \{4\}$

Reflexive Closure :→

If $R \subseteq A \times A$, then reflexive closure of R is $R \cup \Delta$;

where $\Delta = \{(a,a) \mid \forall a \in A\}$

Symmetric Closure :→

If $R \subseteq A \times A$, then the symmetric closure of R is $R \cup R^{-1}$

Ex $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,1), (3,2)\}$

$\Delta = \{(1,1), (2,2), (3,3)\}$

The reflexive closure of R is $R \cup \Delta$

where $R \cup \Delta = \{(1,1) (1,2), (2,1), (2,2), (3,2) (3,3)\}$

The symmetric closure of R is

$R \cup R^{-1} = \{(1,1)\ (1,2)\ ,(2,1)\ ,(3,2)\} \cup \{(1,1), (2,1), (1,2), (2,3)\}$

$= \{(1,1), (1,2), (2,1), (2,3), (3,2)\}$

Note (1) The reflexive closure of any rel^n R is always reflexive.
(2) The symmetric " " " R is " symmetric.

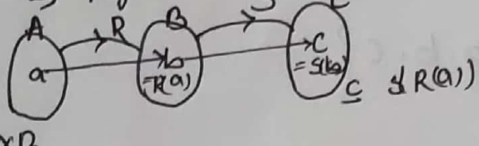
Composition of Relation :→

$R \subseteq A \times A$

$\Delta = \{(a,a) \mid \forall a \in A\}$

$R \cup \Delta \to$ Ref closure

$R \cup R^{-1} \to$ symm. closure.

Let 

$R \subseteq A \times B$

$S \subseteq B \times C$

If $(a,b) \in R$ & $(b,c) \in S$ then $(a,c) \in S o R$.

*) $R o R = R^2$

$R^2 o R = R^3$

*) For $S o R$ we must have

$R(R) = D(S)$

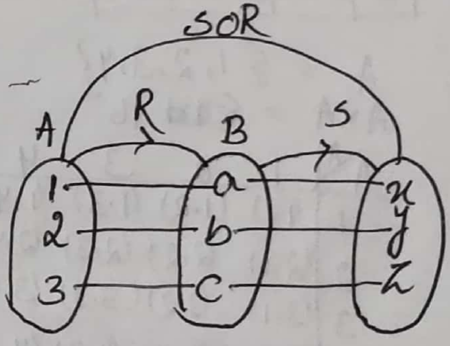
Ex $A = \{1,2,3\}$

$B = \{a,b,c\}$

$C = \{x,y,z\}$

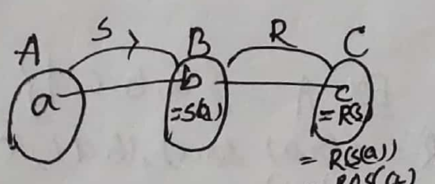
$R \subseteq A \times B$; $R = \{(1,a), (2,b), (2,c)\}$

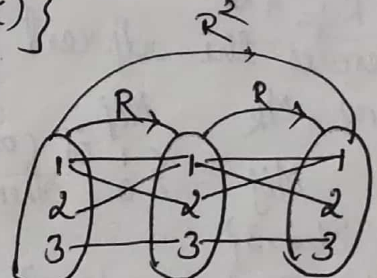
$S \subseteq B \times C$; $R = \{(a,x), (b,y), (c,z)\}$



$S o R \subseteq A \times C$

Here $S o R = \{(1,x), (2,y), (3,z)\}$





Ex Let $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,1), (3,3)\}$

Find $R^2, R^3, R^4, \ldots$

$R^2 = R o R$

$= \{(1,1), (1,2), (2,1), (3,3)\} \, o \, \{(1,1), (1,2), (2,1), (3,3)\}$

$= \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$R^3 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$
$o \, \{(1,1), (1,2), (2,1), (3,3)\}$

$= \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$R^4 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

(60)

Note $R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5 \cdots$

Transitive closure :→

Let $R \subseteq A \times A$

The transitive closure of R is to be denoted by R^*

where $R^* = R \cup R^2 \cup R^3 \cup R^4 \cup \cdots$

Notes :-

*) Reflexive closure of any rel^n^ R is always reflexive.

*) symmetric " " " " " " symmetric.

*) Transitive " " " " " " transitive.

Adjacency Matrix :→

Ex Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

so $|A \times B| = 9$

A \ B	a	b	c
1	(1,a)	(1,b)	(1,c)
2	(2,a)	(2,b)	(2,c)
3	(3,a)	(3,b)	(3,c)

B \ A	1	2	3
a	(a,1)	(a,2)	(a,3)
b	(b,1)	(b,2)	(b,3)
c	(c,1)	(c,2)	(c,3)

Ex $A = \{1, 2, 3, 4\}$

$A \times A = 16$

A \ A	1	2	3	4
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,1)	(4,2)	(4,3)	(4,4)

Note Let $R \subseteq A \times A$

M_R denotes the adjacent matrix of R.

where $M_R = M_{ij}$

$$M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0, & \text{otherwise} \end{cases}$$

Ex $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,1), (3,3)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex $A = \{a, b, c, d\}$

$R = \{(a,a), (a,c), (b,d), (c,c), (d,a), (d,c)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Ex $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ or $A = \{a, b, c\}$

Here $R = \{(a,a), (a,c), (b,b), (c,a), (c,b), (c,c)\}$

Scanned with CamScanner

Warshall's Algorithm (To find Transitive closure)

Find the transitive closure of the relⁿ R where adjacency matrix is $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

step-1

$$W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1,1), (1,2), (2,3), (3,4), (3,5), (4,5)\}$

Col-1	Row-1	Resultings
(1,1)	(1,1) →	(1,1)
	(1,2)	(1,2)

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

step-2

Col.-2	Row-2	Resulting
(1,2)	(2,3) →	(1,3)

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

step-3

Col.-3	Row-3	Resultings
(1,3)	(3,4) →	(1,4)
(2,3)	(3,5)	(1,5)
		(2,4)
		(2,5)

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

step-4

Col.4	row-4	resultings
(1,4)	(4,5)	(1,5)
(2,4)		(2,5)
(3,4)		(3,5)

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

step-5

Col.5	Row 5	Resulting
(1,5)	X	(1,5)
(2,5)		(2,5)
(3,5)		(3,5)
(4,5)		(4,5)

$$W_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

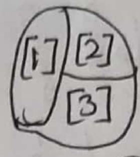
$W_4 = W_5$ be the adjacency matrix of required relⁿ R

transitive closure of R $= \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e)\}$

Scanned with CamScanner

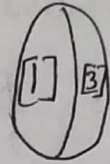
Equivalence class :→

Let R be an equivalence ~~~~ rel^n defined over set A,
$\forall a \in A$, $[a]$ denotes the equivalence class of 'a'
where $[a] = \{b \mid (a,b) \in R\}$



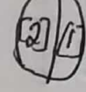
Ex $A = \{1,2,3\}$
$R = \{(1,1), (2,2), (3,3)\}$
So; $[1] = \{1\}$, $[2] = \{2\}$, $[3] = \{3\}$ $\quad [1] \cup [2] \cup [3] = \{1,2,3\} = A$

Ex $A = \{1,2,3\}$
$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$
So; $[1] = \{1,2\}$, $[2] = \{1,2\}$, $[3] = \{3\}$

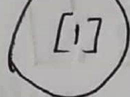


Here $[1] = [2]$
$[1] \cup [3] = \{1,2,3\} = A$

Ex
$R = \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$
$[1] = \{1\}$, $[2] = \{2,3\}$, $[3] = \{2,3\}$
$[2] = [3]$



Ex $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
$[1] = \{1,2,3\}$; $[2] = \{1,2,3\}$; $[3] = \{1,2,3\}$
So $[1] = [2] = [3]$



Notes
→ Since R be an equivalence rel^n, R is reflexive.
So $\forall a \in A$, $(a,a) \in R$
$\Rightarrow a \in [a]$
→ $[a] = [b]$ or $[a] \cap [b] = \phi$
Any equivalence class are either disjoint or identical.
→ $\bigcup_{i=1} [a_i] = A$; ~~classes~~ union of all the distinct equivalence classes is set A.

Ex $R = \{(x,y) \mid x \equiv y \pmod 2 ; \forall x,y \in \mathbb{Z}\}$ be an equivalence rel^n.
find all the possible equivalence classes,

Sol^n we know $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$(x,y) \in R \Leftrightarrow x \equiv y \pmod 2$
i;e $2 \mid y - x$
i;e $y - x = 2K$, $K \in \mathbb{Z}$
i;e $y = x + 2K$
since $[x] = \{y \mid (x,y) \in R\}$
$= \{y \mid y = x + 2K, K \in \mathbb{Z}\}$
$[0] = \{y \mid y = 2K, K \in \mathbb{Z}\} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

$[1] = \{y \mid y = 2k+1,\ k \in \mathbb{Z}\} = \{\cdots -5, -3, -1, 1, 3, 5, 7, 9, \cdots\}$

$[2] = \{y \mid y = 3k+2,\ k \in \mathbb{Z}\} = \{\cdots, -5, -3, -1, 1, 3, 5, 7, 9, \cdots\}$

$[0] = [2] = [4] = [6] = \cdots$

$[1] = [3] = [5] = [7] = \cdots$

$[-1] = \{y \mid y = 2k-1,\ \forall k \in \mathbb{Z}\} = \{\cdots, -7, -5, -3, -1, 1, 3, 5, 7, \cdots\}$

$[-2] = \{y \mid y = 2k-2;\ k \in \mathbb{Z}\} = \{\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots\}$

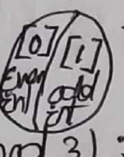
$[-3] = \{y \mid y = 2k-3;\ k \in \mathbb{Z}\} = \{\cdots, -5, -3, -1, 1, 3, 5, \cdots\}$

$[-4] = \{y \mid y = 2k-4;\ k \in \mathbb{Z}\} = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$

clearly $[0] = [\pm 2] = [\pm 4] = [\pm 6] = \cdots = [2m]$

$[1] = [3] = [5] = [7] = \cdots = [-1] = [-3] = [-5] = \cdots = [2m+1]$

So, we have two distinct equivalence classes such as $[0]$ and $[1]$.



Ex $R = \{(x,y) \mid x \equiv y \pmod 3;\ \forall x, y \in \mathbb{Z}\}$ be an equivalence rel^n, find all the possible equivalence classes.

Sol^n $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \cdots\}$

$(x,y) \in R \Leftrightarrow x \equiv y \pmod 3$

i.e $3 \mid y-x$

i.e $y - x = 3k;\ k \in \mathbb{Z}$

since $[x] = \{y \mid (x,y) \in R\}$

$= \{y \mid y = x + 3k,\ k \in \mathbb{Z}\}$

$[0] = \{y \mid y = 3k,\ k \in \mathbb{Z}\} = \{\cdots, -6, -3, 0, 3, 6, 9, \cdots\}$

$[1] = \{y \mid y = 3k+1;\ k \in \mathbb{Z}\} = \{\cdots, -8, -5, -2, 1, 4, 7, 10, \cdots\}$

$[2] = \{y \mid y = 3k+2;\ k \in \mathbb{Z}\} = \{\cdots, -10, -7, -4, -1, 2, 5, 8, 11, \cdots\}$

$[3] = \{y \mid y = 3k+3;\ k \in \mathbb{Z}\} = \{\cdots, -9, -6, -3, 0, 3, 6, 9, 12, \cdots\}$

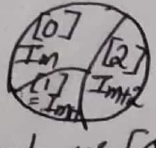
$[4] = \{y \mid y = 3k+4;\ k \in \mathbb{Z}\} = \{\cdots, -8, -5, -2, 1, 4, 7, 10, \cdots\}$

$[5] = \{y \mid y = 3k+5;\ k \in \mathbb{Z}\} = \{\cdots, -7, -4, -1, 2, 5, 8, 11, 14, \cdots\}$

$[0] = [3] = [6] = \cdots = [-3] = [-6] = \cdots$

$[1] = [4] = [7] = \cdots = [-2] = [-5] = \cdots$

$[2] = [5] = [8] = \cdots = [-1] = [-4] = \cdots$



So we have the distinct equivalence classes such as $[0], [1]$ & $[2]$.

* In general the equivalence rel^n congruence modulo 'n' over Z gives 'n' no. of distinct equivalence classes such as $[0], [1], [2], [3], \ldots [n-1]$.

<u>Partition of set</u> :→

Let A be any non-empty set $A_1, A_2, A_3, \ldots, A_n$ be subset of π.

Def^n of partition: The set $\pi = \{A_1, A_2, \ldots, A_k\}$ is said to be partition of set A iff (i) $A_i \cap A_j = \phi$ for $i \neq j$

(ii) $\bigcup_{i=1}^{m} A_i = A$

→ If $\pi = \{A_1, A_2, \ldots, A_k\}$ then each member of π is said to be a Block.

→ Every element of the block corresponds with each element of that block i.e if $A = \{a, b\}$ then $(a,a) \in R$, $(a,b) \in R$ and $(b,b) \in R$

<u>Ex</u> Let $A = \{a, b, c, d, e, f\}$

$A_1 = \{a, b\}$, $A_2 = \{c, d, e\}$, $A_3 = \{f\}$, $A_4 = \{d, e, f\}$, $A_5 = \{a, b, c\}$

(i) Is $\pi = \{A_1, A_2, A_3\}$ be a partition of A?

<u>Ans</u> $A_1 \cap A_2 = \{a, b\} \cap \{c, d, e\} = \phi$

$A_2 \cap A_3 = \{c, d, e\} \cap \{f\} = \phi$

$A_3 \cap A_1 = \{f\} \cap \{a, b\} = \phi$

$A_1 \cup A_2 \cup A_3 = \{a, b\} \cup \{c, d, e\} \cup \{f\} = \{a, b, c, d, e, f\}$.

Thus π be a partition of set A.

Recurrsive Relation :→

* The linear R·R with constant coefficient is of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$$

If $f(x) = 0$ then it is homogeneous otherwise non-homogeneous.

& is of order k.

Ex

$a_n + a_{n-1} = 0$ → Homogeneous and order = 1.

$a_n = a_{n-1} + a_{n-2}$ → " & " = 2.

$a_n - 3a_{n-2} = 0$ → " " , = 2.

$a_n + a_{n-1} - 2a_{n-2} + a_{n-3} = n$ → non-homogeneous, order = 3

Soln of homogeneous linear R·R :→

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$$

Let the trial soln is $a_n = \alpha^n$

so $a_{n-1} = \alpha^{n-1}$, $a_{n-2} = \alpha^{n-2}$ ... $a_{n-k} = \alpha^{n-k}$

Putting in the given R·R

$$\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \cdots + c_k\alpha^{n-k} = 0$$

$$\Rightarrow \alpha^{n-k}\{\alpha^k + c_1\alpha^{k-1} + c_2\alpha^{k-2} + \cdots + c_k\} = 0$$

As $\alpha^{n-k} \neq 0$, $\boxed{\alpha^k + c_1\alpha^{k-1} + c_2\alpha^{k-2} + \cdots + c_k = 0}$

⇓

This is characteristic eqn.

solving this eqn, we will get k-values of 'α' such as $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$.

(1) when the roots are real & distinct

$$a_n^{(h)} = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + \cdots + A_k\alpha_k^n$$

(2) Equal & real roots :-

→ $\alpha_1 = \alpha_2$ → $a_n = (A_1 + A_2 n)\alpha_1^n$

→ $\alpha_1 = \alpha_2$ & $\alpha_3, \alpha_4, \ldots, \alpha_k$ are distinct

$$a_n = (A_1 + A_2 n)\alpha_1^n + A_3\alpha_3^n + A_4\alpha_4^n + \cdots + A_k\alpha_k^n$$

→ $\alpha_1 = \alpha_2 = \alpha_3$ → $a_n = (A_1 + A_2 n + A_3 n^2)\alpha_1^n$

$\rightarrow \alpha_1 = \alpha_2 = \alpha_3$ and others are real & distinct ;

$a_n = (A_1 + A_2 n + A_3 n^2)\alpha_1^n + A_4\alpha_4^n + A_5\alpha_5^n + \cdots + A_k\alpha_k^n$

$\rightarrow \alpha_1 = \alpha_2$ & $\alpha_3 = \alpha_4$

$a_n = (A_1 + A_2 n)\alpha_1^n + (A_3 + A_4 n)\alpha_3^n$

$\rightarrow \alpha_1 = \alpha_2$ & $\alpha_3 = \alpha_4$ but eigther are distinct

$a_n = (A_1 + A_2 n)\alpha_1^n + (A_3 + A_4 n)\alpha_3^n + A_5\alpha_5^n + A_6\alpha_6^n + \cdots + A_k\alpha_k^n$

Ex $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Ans Homogeneous & order $= 2$

$\left[\because a_n = \alpha^n,\ a_{n-1} = \alpha^{n-1},\ a_{n-2} = \alpha^{n-2}\right.$

Putting $\alpha^n - 5\alpha^{n-1} + 6\alpha^{n-2} = 0$

$\Rightarrow \alpha^{n-2}(\alpha^2 - 5\alpha + 6) = 0$

$\left.\Rightarrow \alpha \neq 0 \ \& \ \alpha^2 - 5\alpha + 6 = 0\right]$

eq is $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \boxed{\alpha = 2, 3}$

$a_n = c_1 2^n + c_2 3^n$

Ex $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Sol Homo. & order $= 2$

$\left[\because a_n = \alpha^n\right.$

$a_{n-1} = \alpha^{n-1}$

$a_{n-2} = \alpha^{n-2}$

putting in this given r.r, $\alpha^n - 4\alpha^{n-1} + 4\alpha^{n-2} = 0$

$\Rightarrow \alpha^{n-2}(\alpha^2 - 4\alpha + 4) = 0$

$\left.\alpha^{n-2} \neq 0;\ \alpha^2 - 4\alpha + 4 = 0\right]$

Ch. eqn is $\alpha^2 - 4\alpha + 4 = 0$

$\Rightarrow \boxed{\alpha = 2, 2}$

$a_n = (c_1 + c_2 n)2^n$

Ex $a_n = a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 1$

$a_n - a_{n-1} - a_{n-2} = 0$

Sol Homogeneous, order $= 2$

Ch. eqn is $\alpha^2 - \alpha - 1 = 0$

$\Rightarrow \alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ —— (*)

$a_0 = 1 \Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 1$

$\Rightarrow c_1 + c_2 = 1$ —— (1)

$a_1 = 1 \Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

$\Rightarrow (1+\sqrt{5})c_1 + (1-\sqrt{5})c_2 = 2$ —— (2)

Solving ① & ② we get

$$(1+\sqrt5)c_1 + (1-\sqrt5)c_2 = 2$$
$$(1+\sqrt5)c_1 + (1+\sqrt5)c_2 = 1+\sqrt5$$
$$(-)\quad(-)\quad(-)$$

$$(1-\sqrt5)c_2 - (1+\sqrt5)c_2 = 2-1-\sqrt5$$
$$\Rightarrow -2\sqrt5\, c_2 = 1-\sqrt5 \Rightarrow \boxed{c_2 = \frac{1-\sqrt5}{-2\sqrt5}}$$

From eqⁿ-① ; we get $c_1 + c_2 = 1$

$$\Rightarrow c_1 = 1 - c_2 = 1 + \frac{1-\sqrt5}{2\sqrt5} = \frac{2\sqrt5+1-\sqrt5}{2\sqrt5} = \frac{1+\sqrt5}{2\sqrt5}$$
$$\Rightarrow \boxed{c_1 = \frac{1+\sqrt5}{2\sqrt5}}$$

Thus
$$a_n = \frac{1+\sqrt5}{2\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{1-\sqrt5}{2\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n$$
$$= \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^{n+1} - \left(\frac{1-\sqrt5}{2}\right)^{n+1}\right]$$

Ex. $a_n - 3a_{n-1} = 0$; $a_0 = 1$

Solⁿ $a_n - 3a_{n-1} = 0$

Homo. , order = 1

Characteristic eqⁿ is $\alpha - 3 = 0 \Rightarrow \boxed{\alpha = 3}$

$$\boxed{a_n = c_1 3^n}$$

$a_0 = 1 \Rightarrow c_1 3^0 = 1 \Rightarrow \boxed{c_1 = 1}$

So $a_n = 3^n$

Ex $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

Solⁿ Homo. , order = 3.

Characteristic eqⁿ is $\alpha^3 - 6\alpha^2 + 12\alpha - 8 = 0$
$$\Rightarrow (\alpha-2)^3 = 0 \Rightarrow \boxed{\alpha = 2, 2, 2}$$

$$a_n = (c_1 + c_2 n + c_3 n^2)2^n$$

Ex $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$

Homo. & order = 3.

Ch. eqⁿ is $\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$
$$\Rightarrow \boxed{\alpha = 3, 2, 2}$$

$$a_n = c_1 3^n + (c_2 + c_3 n)2^n.$$

Ex

If 2, 2, 3, 3 be the characteristic roots then find the R.R and its solⁿ $\alpha_1 = 2,\ \alpha_2 = 2,\ \alpha_3 = 3,\ \alpha_4 = 3$

characteristic eqⁿ will be

$$(\alpha-2)^2(\alpha-3)^2 = 0$$
$$\Rightarrow (\alpha^2-4\alpha+4)(\alpha^2-6\alpha+9) = 0$$
$$\Rightarrow \alpha^4-6\alpha^3+9\alpha^2-4\alpha^3+24\alpha^2-36\alpha+4\alpha^2-24\alpha+36 = 0$$
$$\Rightarrow \alpha^4-10\alpha^3+37\alpha^2-60\alpha+36 = 0$$

Note

* polynomial of degree 'K' is of the form
$$f(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + \dots + a_K n^K$$

→ polynomial of degree 0, $f(n)$ = constant.

→ " " " 1, $f(n) = a_0 + a_1 n \,/\, an$

→ " " , 2, $f(n) = a_0 + a_1 n + a_2 n^2 \,/\, an^2 \,/\, an^2+bn \,/\, an^2+c$.

Solⁿ of Linear non-homogeneous Recurrence Relation with constant coefficient :→

* The R.R is of the form
$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_K a_{n-K} = f(n)$$

→ It's general or total solⁿ is on 'a_n' where
$$a_n^{(T)} = a_n^{(h)} + a_n^{(P)}$$

$a_n^{(h)}$ be the homog solⁿ which will be obtained by taking $f(n) = 0$

$a_n^{(P)}$ be the particular solⁿ.

How to find particular solⁿ :→

(1) If $f(n)$ = polynomial of degree K, and 1 is not a ch. root then assume $a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \dots + A_K n^K$

(2) If $f(n)$ = (poly. of deg K) × β^n

(i) where β is not a ch. root then $a_n^{(P)} = (A_0 + A_1 n + A_2 n^2 + \dots + A_K n^K)\beta^n$

(ii) where β is a ch. root of multiplication 'm' thus
$$a_n^{(P)} = (A_0 + A_1 n + A_2 n^2 + \dots + A_K n^K)\beta^n \cdot n^m$$

Scanned with CamScanner

(iv) After assuming 'a_n' find $a_{n-1}, a_{n-2}, a_{n-3}, \dots$ etc.
put in this given R.R;
Comparing both sides to get $A_0, A_1, A_2, \dots, A_k$.

Ex

$$a_n - 5a_{n-1} + 6a_{n-2} = 3$$

To get homogeneous sol^n: →

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

Ch. eq^n is $\alpha^2 - 5\alpha + 6 = 0 \Rightarrow \alpha = 2, 3$

$$a_n^{(h)} = c_1 2^n + c_2 3^n$$

To get particular sol^n: →

Here $f(n) = 3$ = polynomial of deg. 0.
1 is not a ch. root.
So assume $a_n^{(P)} = A_0 \Rightarrow a_{n-1} = A_0, a_{n-2} = A_0$

putting in the given R.R

$$A_0 - 5A_0 + 6A_0 = 3 \Rightarrow 2A_0 = 3 \Rightarrow \boxed{A_0 = 3/2}$$

$$\boxed{a_n^{(P)} = \frac{3}{2}}$$

Required sol^n is $a_n^{(T)} = a_n^{(h)} + a_n^{(P)} = c_1 2^n + c_2 3^n + \frac{3}{2}$

Ex

$$a_n - 2a_{n-1} + a_{n-2} = 5 \quad \text{--- (1)}$$

Sol^n Homogeneous sol^n: → $a_n - 2a_{n-1} + a_{n-2} = 0$

Ch. eq^n is $\alpha^2 - 2\alpha + 1 = 0$
$\Rightarrow \boxed{\alpha = 1, 1}$

$$\boxed{a_n^{(h)} = (c_1 + c_2 n) 1^n = c_1 + c_2 n}$$

Particular sol^n $f(n) = 5 = 5 \times 1 = 5 \times 1^n = (\text{poly. of deg. } 0) \times 1^n$

where 1 is a ch. root of multiplicity 2.

So $a_n^{(P)} = A_0 \cdot 1^n \cdot n^2 = A_0 n^2$

$a_{n-1} = A_0 (n-1)^2$

$a_{n-2} = A_0 (n-2)^2$

putting this in eq^n-① ; we get

$A_0 n^2 - 2A_0(n-1)^2 + A_0(n-2)^2 = 5$

$\Rightarrow A_0 n^2 - 2A_0 n^2 + 4A_0 n - 2A_0 + A_0 n^2 - 4A_0 n + 4A_0 = 5$

$\Rightarrow 2A_0 = 5 \Rightarrow \boxed{A_0 = \frac{5}{2}}$

so $\boxed{a_n^{(P)} = \frac{5}{2} n^2}$

Required sol^n is $a_n^{(T)} = c_1 + c_2 n + \frac{5}{2} n^2$

Ex

$a_n - 5a_{n-1} + 6a_{n-2} = 4^n$ —— ①

Homog. sol^n

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

ch. eq^n is $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \boxed{\alpha = 2, 3}$

$\boxed{a_n^{(h)} = c_1 2^n + c_2 3^n}$

particular sol^n :→

$f(n) = 4^n = 1 \times 4^n$

$= (\text{poly of deg } 0) \times 4^n$

where 4 is not a ch. root

so $a_n^{(P)} = A_0 4^n \Rightarrow a_{n-1} = A_0 4^{n-1}$

$a_{n-2} = A_0 4^{n-2}$

putting eq^n-①

$A_0 4^n - 5A_0 4^{n-1} + 6A_0 4^{n-2} = 4^n$

$\Rightarrow 4^{n-2}\{4^2 A_0 - 5A_0 \cdot 4 + 6A_0\} = 4^n \cdot 4^{-2}$

$\Rightarrow 16A_0 - 20A_0 + 6A_0 = 16$

$\Rightarrow 2A_0 = 16 \Rightarrow \boxed{A_0 = 8}$

$\boxed{a_n^{(P)} = 8 \times 4^n}$

so, $a_n^{(T)} = c_1 2^n + c_2 3^n + 8 \cdot 4^n$.

Ex

$a_n - 5a_{n-1} + 6a_{n-2} = 3^n$

Homog. sol^n $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \boxed{\alpha = 2, 3}$

$\boxed{a_n^{(h)} = c_1 2^n + c_2 3^n}$

Particular sol^n

$f(n) = 3^n = 1 \times 3^n = (\text{poly. of deg. } 0) \times 3^n$

where 3 is a ch. root of multiplication 1.

So $a_n^{(P)} = A_0 \cdot 3^n \cdot n^1$

$\Rightarrow a_{n-1} = A_0 (n-1)\, 3^{n-1}$

$\Rightarrow a_{n-2} = A_0 (n-2) \cdot 3^{n-2}$

Putting, $A_0 \cdot n \cdot 3^n - 5A_0(n-1)3^{n-1} + 6A_0(n-2)\,3^{n-2} = 3^n$

$\Rightarrow 3^2 \cdot A_0 \cdot n - 5A_0(n-1)3 + 6A_0(n-2) = 3^2$

$\Rightarrow 9A_0 n - 15A_0 n + 15A_0 + 6A_0 n - 12A_0 = 9$

$\Rightarrow 3A_0 = 9 \Rightarrow \boxed{A_0 = 3}$

So $\boxed{a_n^{(P)} = 3 \cdot 3^n \times n = n \cdot 3^{n+1}}$

Required sol^n is $a_n^{(T)} = a_n^{(h)} + a_n^{(P)}$

$= c_1 2^n + c_2 3^n + n \cdot 3^{n+1}$

Ex

$a_n - 7a_{n-1} + 10a_{n-2} = n+1$

Homo. sol^n $\quad a_n - 7a_{n-1} + 10a_{n-2} = 0$

Ch. eq^n is $\alpha^2 - 7\alpha + 10 = 0$

$\Rightarrow \alpha = 2, 5$

$a_n^{(h)} = c_1 2^n + c_2 5^n$

Particular sol^n $\quad f(n) = n+1$

$= (\text{poly. of deg. } 1) \times 1^n$

1 is not a ch. root.

So $a_n^{(P)} = A_0 + A_1 n$

$a_{n-1} = A_0 + A_1(n-1)$

$a_{n-2} = A_0 + A_1(n-2)$

putting this value in eq^n we get

$a_n - 7a_{n-1} + 10a_{n-2} = n+1$

Scanned with CamScanner

$\Rightarrow A_0 + A_1 n - 7A_0 - 7A_1(n-1) + 10A_0 + 10A_1(n-2) = n+1$

$\Rightarrow A_0 + A_1 n - 7A_0 - 7A_1 n + 7A_1 + 10A_0 + 10A_1 n - 20A_1 = n+1$

$\Rightarrow 4A_1 n + (4A_0 - 13A_1) = n+1$

Comparing $4A_1 = 1$ & $4A_0 - 13A_1 = 1$

$\Rightarrow \boxed{A_1 = \frac{1}{4}}$ & $4A_0 = 1 + 13A_1 = 1 + \frac{13}{4} = \frac{17}{4}$

$\Rightarrow \boxed{A_0 = \frac{17}{16}}$

So $a_n^{(p)} = \frac{17}{16} + \frac{1}{4}n$

Required sol^n is $a_n^{(T)} = a_n^{(h)} + a_n^{(p)}$

$= c_1 2^n + c_2 5^n + \frac{17}{16} + \frac{1}{4}n$

H.W. $a_n - 7a_{n-1} + 10a_{n-2} = 2n$

Ex $a_n - 5a_{n-1} + 6a_{n-2} = 5^n + n$

Hom. sol^n

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

Ch. eq^n is $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \alpha = 2, 3$

$a_n^{(h)} = c_1 2^n + c_2 3^n$

particular sol^n

$f(n) = 5^n + n$

$= (\text{poly. of deg } 0) \times 5^n + (\text{poly. of deg. } 1) \times 1^n$

Neither 5 nor 1 is a ch. root.

So $a_n^{(p)} = P \cdot 5^n + (An + B)$ where P, A, B are cnst.

$a_{n-1} = P 5^{n-1} + A(n-1) + B$

$a_{n-2} = P \cdot 5^{n-2} + A(n-2) + B$

Putting $P \cdot 5^n + (An+B) - 5\{P \cdot 5^{n-1} + A(n-1) + B\}$

$+ 6\{P \cdot 5^{n-2} + A(n-2) + B\} = 5^n + n$

$\Rightarrow \{P \cdot 5^n - 5 \cdot P \cdot 5^{n-1} + 6 \cdot P \cdot 5^{n-2}\} + \{An + B - 5A(n-1) - 5B + 6A(n-2) + 6B\}$

$= 5^n + n$

Scanned with CamScanner

comparing

$\Rightarrow P\cdot 5^n - P\cdot 5^n + 6\cdot P\cdot 5^{n-2} = 5^n$ & $An+B-5An+5A-5B+6An-12A+6B = n$

$\Rightarrow 6P5^{n-2} = 5^n$ & $2An+(2B-7A) = n$

$\Rightarrow 6P = 5^2$ & $2A=1$ & $2B-7A=0$

$\Rightarrow \boxed{P = \frac{25}{6}}$, $\boxed{A = \frac{1}{2}}$, $B = \frac{7}{2}A = \frac{7}{4}$

so $\boxed{a_n^{(P)} = \frac{25}{6}\cdot 5^n + \left(\frac{1}{2}n + \frac{7}{4}\right)}$

Required sol^n is $a_n^{(T)} = a_n^{(h)} + a_n^{(P)}$

$= c_1 2^n + c_2 3^n + \frac{25}{6}\cdot 5^n + \left(\frac{n}{2} + \frac{7}{4}\right)$

Ex $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$

Sol^n Homogeneous sol^n

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

Ch. eq^n is $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \boxed{\alpha = 2, 3}$

$\therefore \boxed{a_n^{(h)} = c_1 2^n + c_2 3^n}$

Particular sol^n $f(n) = 2^n + n = (\text{poly. of deg } 0)\times 2^n + (\text{poly. of deg } 1)\times 1^n$

where 2 is a ch. root of multiplicity 1.

so $a_n^{(P)} = P\cdot 2^n\cdot n + (An+B)$

$a_{n-1} = P(n-1)\cdot 2^{n-1} + \{A(n-1)+B\}$

$a_{n-2} = P(n-2)\cdot 2^{n-2} + \{A(n-2)+B\}$

putting this

$P\cdot n\cdot 2^n + An + B - 5\{P(n-1)2^{n-1} + A(n-1) + B\} + 6\{P(n-2)\cdot 2^{n-2} + A(n-2) + B\} = 2^n + n$

$\Rightarrow \{P\cdot n\cdot 2^n - 5\cdot P(n-1)\cdot 2^{n-1} + 6P(n-2)\cdot 2^{n-2}\} + \{An + B - 5A(n-1) - 5B + 6A(n-2) + 6B\} = 2^n + n$

$\Rightarrow P n\cdot 2^n - 5P(n-1)2^{n-1} + 6P(n-2)2^{n-2} = 2^n$ & $An+B-5A(n-1)-5B+6A(n-2)+6B = n$

& $An+B-5A(n-1)-5B+6A(n-2)+6B = n$

$\Rightarrow 4Pn - 10P(n-1) + 6P(n-2) = 4$ & $An+B-5An+5A-5B+6An-12A+6B = n$

$\Rightarrow 4Pn - 10Pn + 10P + 6Pn - 12P = 4$ & $2An + 2B - 7A = n$

$\Rightarrow -2P = 4$, $2A = 1$, $2B - 7A = 0$

$\Rightarrow \boxed{P = -2}$, $\boxed{A = \frac{1}{2}}$, $\boxed{B = \frac{7}{4}}$ so $a_n^{(P)} = -2\cdot 2^n\cdot n + \frac{1}{2}n + \frac{7}{4}$

$a_n^{(T)} = a_n^{(h)} + a_n^{(P)}$

Binomial Theorem :→

$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \dots + {}^nC_n a^{n-n} x^n$

General form → ${}^nC_r \, a^{n-r} x^r$

co-efficient of G.F → ${}^nC_r$

In particular ,

$\Rightarrow (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots + x^n$

General form $\frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$

Co-eff. of G.f. $\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$

$\Rightarrow (1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots$

$\Rightarrow (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \dots$

G.F $(-1)^r (r+1) x^r$; Co-eff. of G.F. $(-1)^r (r+1)$

$\Rightarrow (1+x)^{-2} = 1 - 2x + 3x^2 \cancel{+} - 4x^3 + 5x^4 + \dots$

G.F $(-1) x^r (r+1)$; C.G.F $(-1)^r (1+r)$

$\Rightarrow (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$; G.F $(-1) x^r$, C.G.F $(-1)^r$

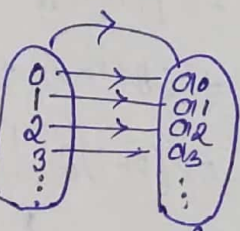
$\Rightarrow (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$; G.F x^r ; C.G.F $(1)^r$

Q $(a-x)^n = ?$ what will be its general term & what is its co-efficient ?

Soln $(a-x)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \dots + (-1)^n \, {}^nC_n x^n$

G.F $(-1)^r \, {}^nC_r \, a^{n-r} . x^r$

C.G.F $(-1)^r \, {}^nC_r$



It is a bijective funⁿ. Its term are $a_0, a_1, a_2, \dots$

* Let 'a_n' be a sequence

* Those terms generates a power series of the form $a_0 + a_1 x + a_2 x^2 + \dots$ If this power series can be converted in term of a funⁿ then that funⁿ is said to be generating function

and to be denoted by $A(x)$.

* For the sequence 'a_n' then G.F. is $A(x)$ where

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Ex (1) $a_n = 1$

Its terms are $1, 1, 1, 1, \ldots$

so $A(x) = 1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + \cdots$

$= 1 + x + x^2 + x^3 + \cdots$

$= \frac{1}{1-x} = (1-x)^{-1}$

(2) $a_n = (-1)^n$

Its terms are $1, -1, 1, -1, 1, -1 \ldots$

$A(x) = 1 - 1\cdot x + 1\cdot x^2 - 1\cdot x^3 + 1\cdot x^4 + \cdots$

$= 1 - x + x^2 - x^3 + x^4 + \cdots$

$= \frac{1}{1+x} = (1+x)^{-1}$

(3) $a_n = a^n$

Its terms are $a^0, a^1, a^2, a^3, a^4, \ldots$

$A(x) = a^0 + a^1 x + a^2 x^2 + a^3 x^3 + \cdots$

$= 1 + ax + (ax)^2 + (ax)^3 + \cdots$

$= \frac{1}{1-ax}$

In particular $2^n \to \frac{1}{1-2x}$

$3^n \to \frac{1}{1-3x}$

$4^n \to \frac{1}{1-4x}$

(4) $a_n = (-a)^n$

Its terms are $(-a)^0, (-a)^1, (-a)^2, (-a)^3 \ldots$

$= 1, -a, a^2, -a^3, a^4, -a^5, \ldots$

$A(x) = 1 - ax + a^2x^2 - a^3x^3 + \cdots$

$= \frac{1}{1+ax}$

since $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$

$\rightarrow \sum_{n=1}^{\infty} a_n x^n = a_1x + a_2x^2 + a_3x^3 + \cdots = A(x) - a_0$

$\rightarrow \sum_{n=2}^{\infty} a_n x^n = a_2x^2 + a_3x^3 + a_4x^4 + \cdots = A(x) - a_0 - a_1x$

$\rightarrow \sum_{n=3}^{\infty} a_n x^n = a_3x^3 + a_4x^4 + a_5x^5 + \cdots = A(x) - a_0 - a_1x - a_2x^2$

$\rightarrow \sum_{n=1}^{\infty} a_{n-1} x^n = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots$

$= x\{a_0 + a_1x + a_2x^2 + \cdots\}$

$= x A(x)$

$\rightarrow \sum_{n=2}^{\infty} a_{n-2} x^n = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \cdots$

$= x^2\{a_0 + a_1x + a_2x^2 + \cdots\}$

$= x^2 A(x)$

$\rightarrow \sum_{n=3}^{\infty} a_{n-3} x^n = a_0x^3 + a_1x^4 + a_2x^5 + \cdots$

$= x^3(a_0 + a_1x + a_2x^2 + \cdots)$

$= x^3 A(x)$

In general $\boxed{\sum_{n=k}^{\infty} a_{n-k} x^n = x^k A(x)}$

$\Rightarrow \sum_{n=2}^{\infty} a_{n-1} x^n = a_1x^2 + a_2x^3 + a_3x^4 + \cdots$

$= x\{a_1x + a_2x^2 + a_3x^3 + \cdots\}$

$= x\{A(x) - a_0\}$

<u>Ex</u> $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Homogeneous with order 2 $(n \geq 2)$

$\sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1}x^n + 6\sum_{n=2}^{\infty} a_{n-2}x^n = 0$

$\Rightarrow A(x) - a_0 - a_1x - 5x\{A(x) - a_0\} + 6x^2 A(x) = 0$

$\Rightarrow (1 - 5x + 6x^2) A(x) = a_0 + a_1x - 5a_0x \Rightarrow A(x) = \dfrac{a_0 + a_1x - 5a_0x}{1 - 5x + 6x^2}$

$= \dfrac{a_0 + a_1x - 5a_0x}{(1-2x)(1-3x)}$

$\Rightarrow A(x) = \dfrac{c_1}{1-2x} + \dfrac{c_2}{1-3x}$, where $c_1(1-3x) + c_2(1-2x) = a_0 + a_1x - 5a_0x$

Required soln will be $a_n = c_1(2^n) + c_2(3^n)$

Ex $a_n - 7a_{n-1} + 10a_{n-2} = 0$; $a_0 = 1, a_1 = 1$

sol^n Homogeneous eq^n, with order = 2 $(n \geq 2)$

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$\Rightarrow A(x) - a_0 - a_1 x - 7x\{A(x) - a_0\} + 10x^2 A(x) = 0$

$\Rightarrow (1 - 7x + 10x^2) A(x) = a_0 + a_1 x - 7a_0 x$ $\Rightarrow (1-2x)(1-5x)A(x) = 1 + x - 7x = 1 - 6x$

$\Rightarrow A(x) = \dfrac{1-6x}{(1-2x)(1-5x)}$

$\left[\because \dfrac{1-6x}{(1-2x)(1-5x)} = \dfrac{c_1}{1-2x} + \dfrac{c_2}{1-5x}\right.$

$\Rightarrow c_1(1-5x) + c_2(1-2x) = 1-6x$

$\Rightarrow (c_1 + c_2) - (5c_1 + 2c_2)x = 1-6x$ $\Rightarrow$ $c_1 + c_2 = 1$, $5c_1 + 2c_2 = 6$

$\left. c_1 = 4/3,\ c_2 = -\frac{1}{3}\right]$

$\Rightarrow A(x) = \dfrac{4/3}{1-2x} - \dfrac{1/3}{1-5x}$

By parts

$a_n = \frac{4}{3}(2^n) - \frac{1}{3}(5^n)$

Ex $a_n - a_{n-1} = n$; $a_0 = 2$

Non-homogeneous, order = 1 $(n \geq 1)$

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} n x^n$$

$= x + 2x^2 + 3x^3 + 4x^4 + \cdots$

$\Rightarrow A(x) - a_0 - x A(x) = x + 2x^2 + 3x^3 + 4x^4 + \cdots$

$\Rightarrow (1-x)A(x) = a_0 + x\{1 + 2x + 3x^2 + 4x^3 + \cdots\}$

$= 2 + x\dfrac{1}{(1-x)^2}$

$A(x) = \dfrac{2}{1-x} + \dfrac{x}{(1-x)^3}$

$= \dfrac{2}{1-x} - \dfrac{1}{(1-x)^2} + \dfrac{1}{(1-x)^3}$

So $a_n = 2(1^n) - (n+1)$

$*\ \dfrac{2}{1-x} + \dfrac{x}{(1-x)^3} = \dfrac{2(1-x)^2 + x}{(1-x)^3}$

$= \dfrac{2 - 4x + 2x^2 + x}{(1-x)^3} = \dfrac{2 - 3x + 2x^2}{(1-x)^3}$

Consider $\dfrac{2-3x+2x^2}{(1-x)^3} = \dfrac{c_1}{1-x} + \dfrac{c_2}{(1-x)^2} + \dfrac{c_3}{(1-x)^3}$

$\Rightarrow c_1(1-x)^2 + c_2(1-x) + c_3 = 2x^2 - 3x + 2$

$\Rightarrow c_1(1 + x^2 - 2x) + c_2(1-x) + c_3 = 2x^2 - 3x + 2$

$\Rightarrow c_1 + c_1 x^2 - 2c_1 x + c_2 - c_2 x + c_3 = 2x^2 - 3x + 2$

$\Rightarrow c_1 x^2 + (-2c_1 - c_2)x + (c_1 + c_2 + c_3) = 2x^2 - 3x + 2$

$\Rightarrow \boxed{c_1 = 2}$, $2c_1 + c_2 = 3$ $\Rightarrow \boxed{c_2 = -1}$, $c_1 + c_2 + c_3 = 2$ $\Rightarrow \boxed{c_3 = 1}$

$\Rightarrow \dfrac{2-3x+2x^2}{(1-x)^3} = \dfrac{2}{1-x} + \dfrac{(-1)}{(1-x)^2} + \dfrac{1}{(1-x)^3}$

Ex $a_n - 3a_{n-1} + 2a_{n-2} = 4^n$; $a_0 = 1, a_1 = 1$

Non-homogeneous order = 2

$$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{n=2}^{\infty} a_{n-1}x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2}^{\infty} 4^n x^n$$

$\Rightarrow (A(x) - a_0 - a_1 x) - 3x\{A(x) - a_0\} + 2x^2 A(x) = 4^2x^2 + 4^3x^3 + 4^4x^4 + \cdots$

$\Rightarrow (1 - 3x + 2x^2)A(x) - a_0 - a_1 x + 3a_0 x = (4x)^2 + (4x)^3 + \cdots$

$\Rightarrow (1 - 3x + 2x^2)A(x) - 1 - x + 3x = (4x)^2\{1 + 4x + (4x)^2 + \cdots\}$

$\Rightarrow (1 - 3x + 2x^2)A(x) - 1 + 2x = 16x^2 \cdot \dfrac{1}{1-4x}$

$$\Rightarrow (1-3x+2x^2)A(x) = 1-2x+\frac{16x^2}{1-4x}$$

$$= \frac{(1-2x)(1-4x)+16x^2}{1-4x} = \frac{1-6x+8x^2+16x^2}{1-4x}$$

$$= \frac{1-6x+24x^2}{1-4x}$$

$$\Rightarrow A(x) = \frac{(1-6x+24x^2)}{(1-4x)(1-3x+2x^2)}$$

$$\Rightarrow A(x) = \frac{1-6x+24x^2}{(1-4x)(1-2x)(1-x)}$$

$$\left[\frac{1-6x+24x^2}{(1-4x)(1-2x)(1-x)} = \frac{c_1}{1-4x} + \frac{c_2}{1-2x} + \frac{c_3}{1-x} \right.$$

$$c_1(1-2x)(1-x) + c_2(1-4x)(1-x) + c_3(1-4x)(1-2x) = 1-6x+24x^2$$

$$\Rightarrow c_1(1-2x)(1-x) + c_2(1-4x)(1-x) + c_3(1-6x+8x^2) = 1-6x+24x^2$$

$$\Rightarrow c_1(1-3x+2x^2) + c_2(1+4x^2-5x) + c_3(1-6x+8x^2) \ldots (2c_1+4c_2+8c_3) = 1-6x+24x^2$$

$$\Rightarrow (c_1+c_2+c_3) - (3c_1+5c_2+6c_3)x + (2c_1+4c_2+8c_3) = 1-6x+24x^2$$

$$\Rightarrow \left.\begin{array}{l} c_1+c_2+c_3 = 1 \\ 3c_1+5c_2+6c_3 = 6 \\ 2c_1+4c_2+8c_3 = 24 \end{array}\right] \quad \left.\begin{array}{l} c_1 = 8/3 \\ c_2 = -8 \\ c_3 = 19/3 \end{array}\right. \text{(by Calculator)} \Big]$$

$$\Rightarrow A(x) = \frac{1-6x+24x^2}{(1-4x)(1-2x)(1-x)} = \frac{8/3}{1-4x} + \frac{(-8)}{1-2x} + \frac{(19/3)}{1-x}$$

Required solⁿ is

$$\boxed{a_n = \frac{8}{3}(4^n) - 8(2^n) + \frac{19}{3}(1)^n}$$

Ans

Graph Theory

Let $G = (V, E)$ be a graph ; where 'V' be the non-empty set called as vertex set.

'E' be the non-empty set called as edge set.

*) The total no. of vertex contained in a graph is its order. i,e order of $G = |V(G)|$

*) The total no. of edges contained in a graph is its size. i;e size of $G = |E(G)|$

*) Generally there are two types of edges

(i) Directed type.

(ii) Non-directed edge.

*) The directed edge is of the form (a, b) $a \rightarrow b$

i;e $e = (a, b)$

Here the edge is from vertex 'a' to vertex 'b'.

*) The non-directed edge is of the form $\{a, b\}$

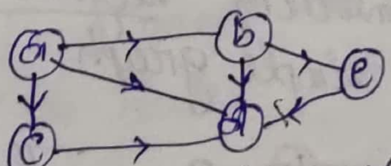
i;e $e = \{a, b\}$

Here the edge is in between the vertex 'a' & 'b'.

*) If all the edges contained in a graph are directed then the graph is said to be directed graph / digraph.

*) If all the edges contained in a graph are non-directed graph is said to be non-directed graph.

Ex (1) G :

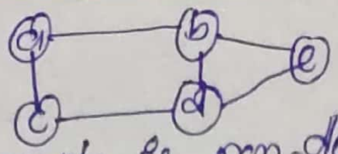


This is a directed graph ; $V = \{a, b, c, d, e\}$

So, order of the graph $= 5$.

$E = \{(a,b), (a,c), (a,d), (c,d), (b,d), (b,e), (e,d)\}$

Size of the graph $= 7$

(2) G = 

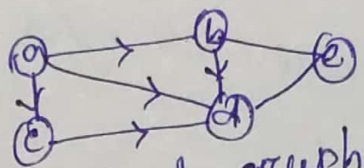
Here the graph is non-directed.

$V = \{a, b, c, d, e\}$ i.e order of $G = 5$

$E = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}\ \{b,d\}, \{c,d\}, \{d,e\}\}$

i.e size of $G = 8$

*) If some of the edges of a graph are directed and the remaining edges are non-directed then the graph is said to be mixed graph.

Ex G : 

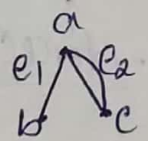
This is a mixed graph.

*) Since each edge have two end vertices and when those two end vertices become same, this edge become a self loop.

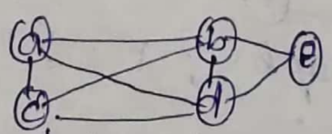
Ex G :  Here self loop exists as vertex 'a'.

That edge is $\{a, a\}$

*) If there exists two or more edge in bet^n the same end vertices then those edges are said to be parallel edge.

Ex  Here e_1 & e_2 are parallel edges.

*) The graph containing neither self loop nor parallel edges is said to a simple graph.

Ex  This is a simple graph.

Complete graph :→

*) A complete graph is a simple graph where each pair of distinct vertices are joined by an edge.

*) Generally the complete graph with n-vertices is denoted by K_n.

*) If a complete graph contain n - vertices then there will be $\frac{n(n-1)}{2}$ no. of edges.

Ex

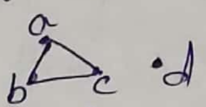
	edge	size
K_1 : .	$n=1$	$\frac{1(1-1)}{2}=0$
K_2 : —	$n=2$	$\frac{2(2-1)}{2}=1$
K_3 : △	$n=3$	$\frac{3(3-1)}{2}=3$
K_4 : ⊠	$n=4$	$\frac{4(4-1)}{2}=6$

Ex If a complete graph contain 10 vertices then there will be how many edges.
or If the order of a complete graph is 10 then what will be its edges.

Ans Here $n=10$
No. of edges $= \frac{n(n-1)}{2} = \frac{10(10-1)}{2} = 45$

*) A graph without any edge is said to be empty or trivial graph. Ex K_1

*) A vertex is said to be an isolated vertex if it is not the end vertex.

Ex 
Here 'd' is the isolated vertex.

* A vertex is said to be pendant vertex if it is the end vertex of exactly one edge. Ex a — b (both 'a' & 'b' are pendant vertices)

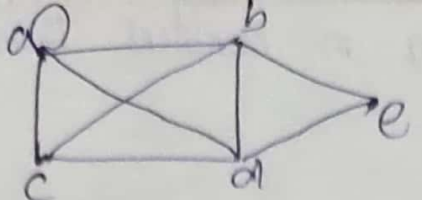
Degree of Vertex :→

An edge 'e' of a graph 'G' is said to incident with the vertex 'v'; if the vertex 'v' is an end vertex of the edge 'e'.

[That means the vertex is incident with 3 edges ⊠ → degree = 3]

⇒ 2 edges 'e' and 'f' which are incident with the common vertex 'v' are said to be adjacent, so the degree of a vertex is the no. of edges incident with that vertex is the self loop have both ends identical, so the degree of a self loop is to be 2:

Ex



Here $d(a) = 5$
$d(b) = 4$
$d(c) = 3$
$d(d) = 4$
$d(e) = 2$

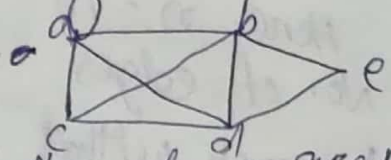
Here order $= 5$
size $= 9$

$\sum deg(v_i) = 5+4+3+4+2 = 18 = 2\times 9$
$= 2 \times$ size of G.

*) Generally the degree of a vertex 'v' is denoted by $d(v)$ or $deg(v)$.

*) Let G be a non-directed graph with vertex $v_1, v_2, \ldots, v_n$ $d_1, d_2, d_3, \ldots, d_n$ be the degree of those vertex resp. Those degree is said to be a degree sequence of their either descending or ascending order.

Ex The degree sequence of the



Here the degree sequence of the above graph is 5, 4, 3, 2.

First theorem of Graph Theory or Handshaking theorem :→

In a non-directed graph $G = (V, E)$

$$\sum_{v_i \in V} deg(v_i) = 2|E|$$

i.e sum of the degree of the vertex $= 2\times$ size of the graph.

* Each edge since it has end vertex contributes precisely to the sum of the degree that means when we are adding the degree of the vertex the edge once counted twice.

Def Any vertex is said to be odd if its degree is odd.
Any vertex is " " " even " " " is even.

Corollary :→

In a non-directed graph there are even no. of odd vertices.

Proof since $G = (V, E)$ be the graph

Let's partition the vertex set V into two disjoint sets U & W $[U \cup W = V,\ U \cap W = \phi]$ in such a that the set U contains all the odd vertex.

& W " " " even " .

By handshaking theorem, we have

$$\sum_{v_i \in V} \deg(v_i) = 2|E|$$

$$\Rightarrow \sum_{v_i \in U} \deg(v_i) + \sum_{v_i \in W} \deg(v_i) = 2|E|$$

$$\Rightarrow \sum_{v_i \in V} \deg(v_i) + \text{Even} = \text{Even}$$

$$\Rightarrow \sum_{v_i \in V} \deg(v_i) = \text{Even} - \text{Even} = \text{Even}$$

$\Rightarrow$ of odd is even.

since sum of even no. of odd is even.

So the no. of odd vertices are even.

Ex Can you draw graph with the following sequences?

4, 4, 3, 3, 3, 2, 2

Ans No ; **Reason** since there are odd no. of odd vertices.

Havel-Hakim Theorem :→

The degree sequence $s, t_1, t_2, t_3, \ldots, t_s, d_1, d_2, \ldots, d_n$ is graphical iff $t_1 - 1, t_2 - 1, t_3 - 1, \ldots, t_s - 1, d_1, d_2, \ldots, d_n$ is graphical.

*) A degree sequence $d_1, d_2, \ldots, d_n$ is graphic / graphical i;e we can draw a simple non-directed graph using the given degree sequence.

Ex 6, 6, 6, 6, 4, 3, 3, 0. Is it graphical?

Ans 6, 6, 6, 6, 4, 3, 3, 0

5, 5, 5, 3, 2, 2, 0

4, 4, 2, 1, 1, 0

3, 1, 0, 0, 0

0, −1, −1, 0

since there ~~are~~ is no vertex with negative degree, so given degree sequence is not graphical.

Ex

6, 5, 5, 4, 3, 3, 2, 2, 2
4, 4, 3, 2, 2, 1, 2, 2

→ 4, 4, 3, 2, 2, 2, 2, 1
3 2 1 1 2, 2, 1

→ 3, 2, 2, 2, 1, 1, 1
1, 1, 1, 1, 1, 1

There exist a graph with given degree sequence

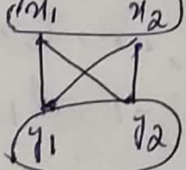
Regular graph :→

The degree of each vertex of a non-directed is equal to 'K' then the graph is said to be a K-regular graph.

Ex (i)  1 - regular
(ii)  2 - regular
(iii)  3 - regular

Biparticle Graph :→

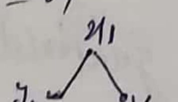
The graph $G = (V, E)$ is said to be a biparticle graph iff the vertex set V is partitioned in to two sets X and Y where $V = X \cup Y$ & $X \cap Y = \phi$ such that end of each edge is in X and the other end is in Y.

Ex :- 

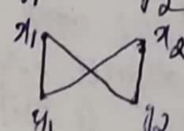
$X = \{x_1, x_2\}$
$Y = \{y_1, y_2\}$

*) If the biparticle graph is simple then it is said to be Complete biparticle graph.

A couple biparticle graph is denoted by $K_{m,n}$ where $|X| = m$, $|Y| = n$

Ex (i) K_{12} : 

$X = \{x_1\}$
$Y = \{y_1, y_2\}$

(ii) K_{22} : 

$X = \{x_1, x_2\}$
$Y = \{y_1, y_2\}$

Graph isomorphism :→

Let G_1 & G_2 are two graphs with vertex set V_1 & V_2 and edge set E_1 & E_2 resp.
Both graphs are said to be isomorphic iff

(i) Both have same order i.e $|V_1| = |V_2|$

(ii) " " " size i.e $|E_1| = |E_2|$

(iii) Both have same degree sequence

(iv) There should exist a bijective funⁿ

$f : V_1 \to V_2$

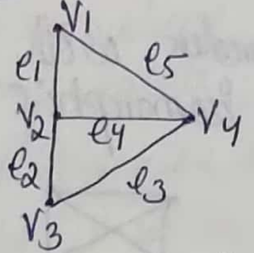
such that if $\{v_1, v_2\} \in E_1$ then

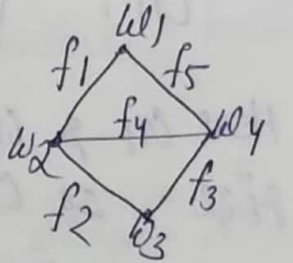
$\{f(v_1), f(v_2)\} \in E_2$.

Ex G_1 :  , G_2 : 

whether both graph are isomorphic ?

Verification :→

G_1 : 

G_2 : 

(i) $V_1 = \{v_1, v_2, v_3, v_4\}$

$V_2 = \{w_1, w_2, w_3, w_4\}$

so $|V_1| = |V_2| = 4$ ∴ Both have same order.

(ii) $E_1 = \{e_1, e_2, e_3, e_4, e_5\}$

$E_2 = \{f_1, f_2, f_3, f_4, f_5\}$

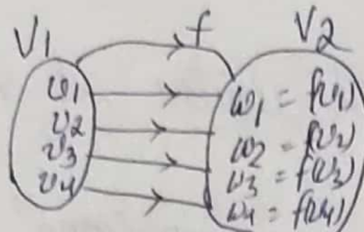
$|E_1| = 5 = |E_2|$ ∴ Both have same size.

(iii) degree of the vertices of

G_1 : $2, 3, 2, 3 \equiv 3, 3, 2, 2$

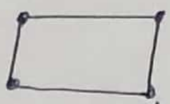
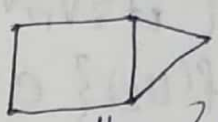
G_2 : $2, 3, 2, 3 \equiv 3, 3, 2, 2$

Both have same degree sequence.

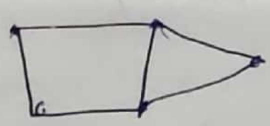
(i) 

$e_1 = \{v_1, v_2\} \in E_1 \Rightarrow \{f(v_1), f(v_2)\} = \{w_1, w_2\} = f_1 \in E_2$
$e_2 = \{v_2, v_3\} \in E_1 \Rightarrow \{f(v_2), f(v_3)\} = \{w_2, w_3\} = f_2 \in E_2$
$e_3 = \{v_3, v_4\} \in E_1 \Rightarrow \{f(v_3), f(v_4)\} = \{w_3, w_4\} = f_3 \in E_2$
$e_4 = \{v_2, v_4\} \in E_1 \Rightarrow \{f(v_2), f(v_4)\} = \{w_2, w_4\} = f_4 \in E_2$
$e_5 = \{v_1, v_4\} \in E_1 \Rightarrow \{f(v_1), f(v_4)\} = \{w_1, w_4\} = f_5 \in E_2$

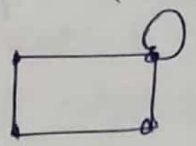
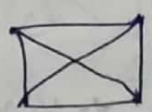
Thus both graph are isomorphic.

Ex G_1:  G_2: 

G_1 & G_2 are not isomorphic as they have different order. $V(G_1) = 4$, $V(G_2) = 5$.

Ex G_1:  G_2: 

Here G_1 & G_2 are same order with different size. so G_1 & G_2 are not isomorphic.

Ex G_1:  G_2: 

deg G_1: = 3, 2, 3, 3 deg G_2 = 3, 3, 3, 3

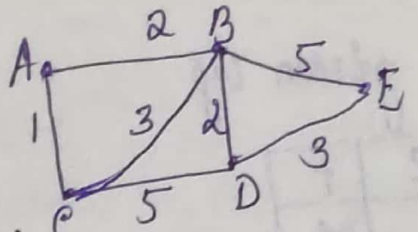
Both have same order, same size but degree of a sequence are different. so G_1 & G_2 are not isomorphic.

Weighted graph: $\longrightarrow$

→ A graph G is said to be weighted if all of its edges are assigned by some real values.

→ If 'e' be the edge then $w(e)$ denotes the weight of the edge 'e'.

EX



$w(\{AC\}) = 1$
$w(\{AB\}) = 2$
$w(\{BC\}) = 3$
$w(\{BD\}) = 2$
$w(\{CD\}) = 5$
$w(\{BE\}) = 5$
$w(\{DE\}) = 3$

* If 'S' be the initial vertex & $V \neq S$, $\lambda(V)$ be the weight of shortest path of vertex V for the in

Shortest path problems :- (Dijkstra's Algorithm)

Input : A connected weighted graph.

output : $\lambda(z)$, the length of shortest distance from a to z.

step-1 : set $\lambda(a) = 0$ & for all vertices $v \neq a$, $\lambda(v) = \alpha$.
set $T = V$, where T = set of vertices having temporary labels.
& V = vertex set of G.

step-2 : Let 'U' be a vertex in T for which $\lambda(U)$ is minimum & hence the permanent label of U.

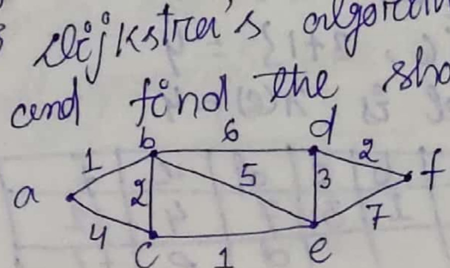
step-3 : If $U = z$ stop.

step-4 : For every edge $e = (U, v)$; incident with U, if $v \in T$, change $\lambda(v)$ to min. $(\lambda(v), \lambda(U) + w(e))$
i.e $\boxed{\lambda(v) = \min.\{\text{old } \lambda(v),\ \lambda(U) + w(e)\}}$

step-5 change T of $T - \{U\}$ and goto step-2.

Examples

Apply Dijkstra's algorithm to the graph given below and find the shortest path from a to f.



Solution The initial labelling is given by

vertex v	a	b	c	d	e	f
$\lambda(v)$	0	∞	∞	∞	∞	∞
T	$\{a$	b	c	d	e	$f\}$

Iteration - 1

$v = a$ has $\lambda(a) = 0$, T becomes $T - \{a\}$.
There are two edges incident with a i;e ab and ac where $b \& c \in T$.

$\lambda(b) = \min\{\text{old } \lambda(b), \lambda(a) + w(ab)\} = \min\{\infty, 0+1\} = 1$

$\lambda(c) = \min\{\text{old } \lambda(c), \lambda(a) + w(ac)\} = \min\{\infty, 0+4\} = 4$

Here min. label is $\lambda(b) = 1$;

vertex	a	b	c	d	e	f
$\lambda(v)$	0	1	4	∞	∞	∞
T	$\{$	b	c	d	e	$f\}$

Iteration - 2

$v = b$, the permanent label of b is 1. T becomes $T - \{b\}$
there are three edges incident with b i;e

$\lambda(c) = \min.\{\text{old } \lambda(c), \lambda(b) + w(bc)\}$
$= \min.\{4, 1+2\} = 3$

$\lambda(d) = \min.\{\text{old } \lambda(d), \lambda(b) + w(bd)\}$
$= \min.\{\infty, 1+6\} = 7$

$\lambda(e) = \min.\{\text{old } \lambda(e), \lambda(b) + w(be)\}$
$= \min.\{\infty, 1+5\} = 6$

Vertex (v)	a	b	c	d	e	f
$\lambda(v)$	0	1	3	7	6	∞
T	$\{$		c	d	e	$f\}$

Here min. label is $\lambda(c) = 3$

Iteration - 3

$v = c$, the permanent label of c is 3, T becomes $T - \{c\}$; there is one edge incident with c i;e ce where $e \in T$.

$\lambda(c) = \min.\{\text{old } \lambda(e), \lambda(c) + w(ce)\}$
$= \min.\{6, 3+1\} = 4$

Thus, min. label is $\lambda(e) = 4$

Vertex (v)	a	b	c	d	e	f
$\lambda(v)$	0	1	3	7	4	11
T	$\{$			d	e	$f\}$

Iteration-4 $u = e$, the permanent label of 'e' is 4, T becomes $T - \{e\}$. There are two edges incident with e i;e ed & ef where $d, f \in T$.

$$\lambda(d) = \min\{\text{old } \lambda(d), \lambda(e) + w(ed)\}$$
$$= \min\{7, 4+3\} = 7$$
$$\lambda(f) = \min\{\text{old } \lambda(f), \lambda(e) + w(ef)\}$$
$$= \min\{\infty, 4+7\} = 11$$

Vertex V	a	b	c	d	e	f
$\lambda(V)$	0	1	3	7	4	11
T	{				d	f}

The minimal label is $\lambda(d) = 7$

Iteration -5 $u = d$, the permanent label of d is 7. T becomes $T - \{d\}$. There is one edge incident with d i;e df where $f \in T$.

$$\lambda(f) = \min\{\text{old } \lambda(f), \lambda(d) + w(df)\}$$
$$= \min\{11, 7+2\} = 9$$

Vertex (V)	a	b	c	d	e	f
$\lambda(V)$	0	1	3	7	4	9
T	{					f}

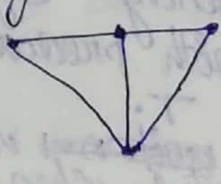
The min. label is $\lambda(f) = 9$

since $u = f$, the only choice, iteration stops. Thus the shortest distance betn a & f is 9 & the shortest path is (a, b, c, d, e, f).

Spanning Tree :→

A subgraph T of a connected graph $G(V, E)$ is called a spanning tree if T is a tree and if T includes every vertex of G i;e $V(T) = V(G)$. If $|V| = n$ & $|E| = m$ then the spanning tree of G must have n vertices & hence 'n-1' edges. We must remove $m-(n-1)$ edges from G to obtain a spanning tree.

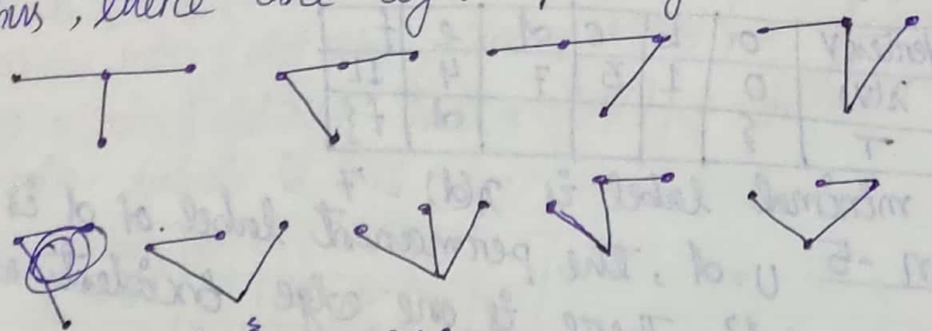
Ex :- Find all spanning tree of the graph G.



Sol^n The graph G has four vertices & hence each spanning tree must have $4-1=3$ edges.
Thus each tree ~~must have 4-1 = 3 edges~~ can be obtained by deleting two of the five edges of G.

So $^5C_2 = \frac{5!}{2!\,3!} = 10$ ways, except that two of the ways lead to disconnected graphs.

Thus, there are eight spanning tree.



Q.5 **Minimal spanning Trees:-**

Let G be a connected weighted graph. The weight of a spanning tree of G is the sum of the weights of the edges. "A minimal spanning Tree of G is a spanning tree of G with minimum weight".

Kruskal's Algorithm :- (To find minimal spanning tree)

This algorithm provides an acyclic subgraph T of a connected weighted graph G which is a minimal spanning tree of G. The algorithm includes the following steps :-

Input : A connected weighted graph G.

Output : A minimal spanning tree T.

step-1 List all the edges (which do not form a loop) of G in non-decreasing order of their weights.

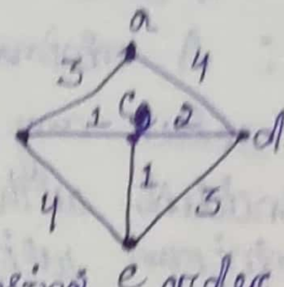
step-2 Select an edge of minimum weight (If more than one edge of minimum weight, arbitrarily choose one of them). This is the first edge of T.

step-3 At each stage, select an edge of minimum weight from all the remaining edges of G if it does not form a circuit with previously selected edges in T. Add the edge to T.

Step-4 Repeat step-3 ~~until~~ until (n-1) edges have been selected, when n is the no. of vertices in G.

Examples

Show how Kruskal's algorithm find a minimal spanning tree for the graph:

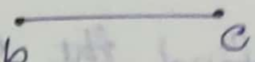


Solution

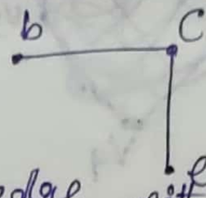
step-1 list the edges in non-decreasing order of their weights:

Edge :	(b,c)	(c,e)	(c,d)	(a,b)	(e,d)	(a,d)	(b,e)
weight :	1	1	2	3	3	4	4

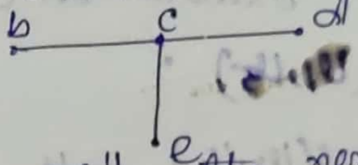
step 2 select the edge (b,c) since it has the smallest weight, include it in T.



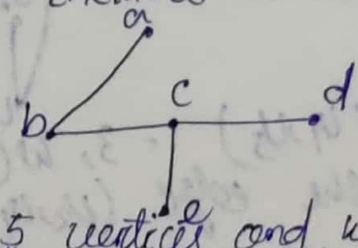
step-3 select an edge with the next smallest weight (c,e), since it does not form circuit with the existing edges in T, include it in T.



step 4 select an edge with the next smallest (c,d) since it does not form circuit with the existing edges in T. so include it in T.



step-5 select an edge with the next smallest weight (a,b) since it does not form circuit with the existing edges in T. so include it in T.



since G contains 5 vertices and we have choosen 4 edges, we stop the algorithm & the minimal spanning tree is produced.

Q.5 Prim's Algorithm :→ (To find minimal spanning tree)

input : A connected weighted graph G.

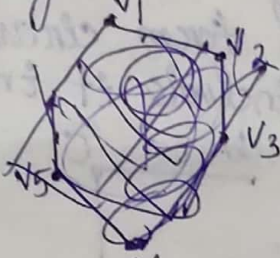
output : A minimal spanning tree T.

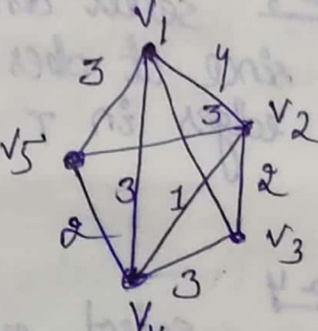
step-1 : select any vertex in G. Among all the edges incident with the selected vertex, choose an edge of minimum weight. include it in T.

step-2 : At each stage, choose an edge of smallest weight joining a vertex already included in T & a vertex not yet included, if it does not form a circuit with edges in T. include it in T.

step-3 Repeat until all the vertices of G are included.

Examples :→ find the minimal spanning tree of the weighted graph using prim's algorithm

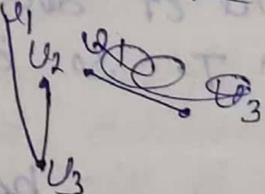


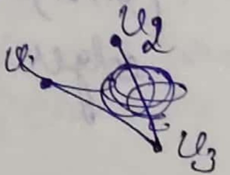


Sol^n

step-1 We choose the vertex v_1, Now edge with smallest weight incident on v_1 is (v_1, v_3). So we chose the edge (v_1, v_3).

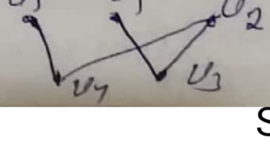
step-2 Now $w(v_1,v_2) = 4$, $w(v_1,v_4) = 3$, $w(v_1,v_5) = 3$, $w(v_3,v_2) = 2$ & $w(v_3,v_4) = 3$. We chose the edge (v_3, v_2) since it is min.





step-3 Again $w(v_1,v_5) = 3$, $w(v_2,v_4) = 1$ & $w(v_3,v_4) = 3$, we choose the edge (v_2, v_4).

step-4 Now we choose the edge (v_4, v_5). Now all the vertices are covered. The minimal spanning tree is produced.



Scanned with CamScanner